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Optimization of the Synchronization of Traffic Lights
NON CONFIDENTIAL

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Contents

Abstract

Le développement des métropoles a considérablement agrandi le parc de véhicules urbains, provoquant de nombreuses inefficacités dans les transports. Les feux de signalisation sont responsables de temps d'attente importants à cause de leur interruptions brutales du trafic. Les coordonner est un moyen de limiter leur impact sur le temps de parcours. Il est important de modèliser le trafic urbain et l'influence des feux de signalisation sur celui-ci. Ce travail étudie la pertinence d'un modèle d'approximation où les flots sortants de l'intersection sont projetés sur un sous-espace, afin de maintenir le nombre de paramètres représentant le trafic constant. Ce modèle a été mis en oeuvre pour la synchronisation de deux puis d'une série de feux de signalisation. La propagation d'une onde de voitures passant toutes les intersections au feu vert s'est revelée optimale dans le cadre de cette étude. Plus que de retrouver un resultat connu, ce modèle facilite analytiquement la transition entre l'étude d'une intersection et d'une série d'intersections. Ce modèle de flots peut être généralisé à d'autres problèmes de trafic urbain tel que l'estimation d'un temps de parcours traversant une série de feux de signalisation.

Global urbanization creates a lot of inefficiencies in urban transportation, by hugely increasing the number of cars in the major cities. Traffic lights are responsible for important delays in arterial traffic because of their behaviour disrupting the traffic flow. Travel times could be reduced by synchronizing signals. To solve this problem, it is necessary to model efficiently the traffic flow. This report presents a model of a projective intersection. The output flow is projected on a subspace to keep constant the numbers of parameters characterizing traffic flows. This model has been implemented to coordinate two and several intersections. It showed a green wave - where all vehicles go through the intersections during the green time - is optimal. In addition to this known result, the transition between the study of one and several intersection is easy thanks to the model. Such a model of traffic flow is general and can be implemented for other situations such as travel time estimation.

Chapter 1

Preliminaries

Since the increase of the number of personal vehicles, traffic jams and inefficiencies are common in the major cities. Even though it is hard to measure its real impact, traffic congestion is the cause of a lot of air pollution. Moreover, according to the Victoria Transport Policy Institute [8], congestion cost is about 0.13 US$ per vehicle mile for an average car. The Texas Transportation Institute [13] estimates that American car drivers have lost 4.2 billion hours in 2007 because of congestion.

These inefficiencies must be reduced by improving arterial traffic management. Urban traffic flows are hard to model because of the huge number of parameters it needs to be described and because of the intrinsic complexity of a network. As traffic lights cause part of the congestions, the model of traffic flow presented in this work is designed to study the evolution of traffic flows through signalized intersections.

1.1 Fundamental diagram

In this work, we will use common assumptions about traffic flow theory. The traffic flow is represented by macroscopic variables of flow $q(x,t)$, density $\rho(x,t)$ and velocity $v(x,t)$. Like in mechanics, we have the relation coming from the definition of flow [7, 10]:

$$\forall x, t \in \mathbb{R}, q(x,t) = \rho(x,t)v(x,t) \tag{1.1}$$

For low values of density, the velocity remains constant and does not depend on flow and density. This velocity is called free flow speed v_f. Nevertheless, after the density reaches a particular value, the critical density ρ_c, vehicles cannot drive at the free flow speed anymore and the flow thus decreases from its maximal value q_{max} until zero reached at the maximal density ρ_{max}.

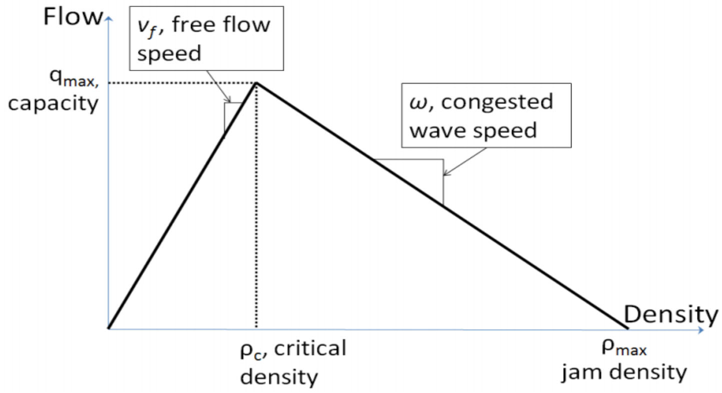


Figure 1.1: The triangular fundamental diagram

The maximal density can be thought physically as the maximum amount of vehicles that can fit in the road. All these assumptions derived from experimental data are summarized in the fundamental diagram *(FD)* [2] illustrated in the figure 1.1, which is in this model assumed to be triangular. This assumption makes the right part of the FD linear with a slope ω corresponding to the congested wave speed.

1.2 Implementation to signalized intersections

In the following, we will apply these assumptions to model traffic flows at intersections equipped with traffic lights.

The California Center for Innovative Transportation [1] has been able to study an intersection using the fundamental diagram and to represent vehicles trajectories on a space time diagram like the one in the figure 1.2. Two discrete regimes appear in this model, the undersaturated one in which some vehicles go through the intersection without stopping and the congested one in which every vehicle stops at least once before going through the intersection.

The interesting fact in this diagram we will focus on in the following is the structure of the output flow. We have three different platoons, one without any vehicles during the red time, one with a flow at critical density ρ_c during the queue discharge lasting a time θ_c, and one at the arriving density during the extra green time.

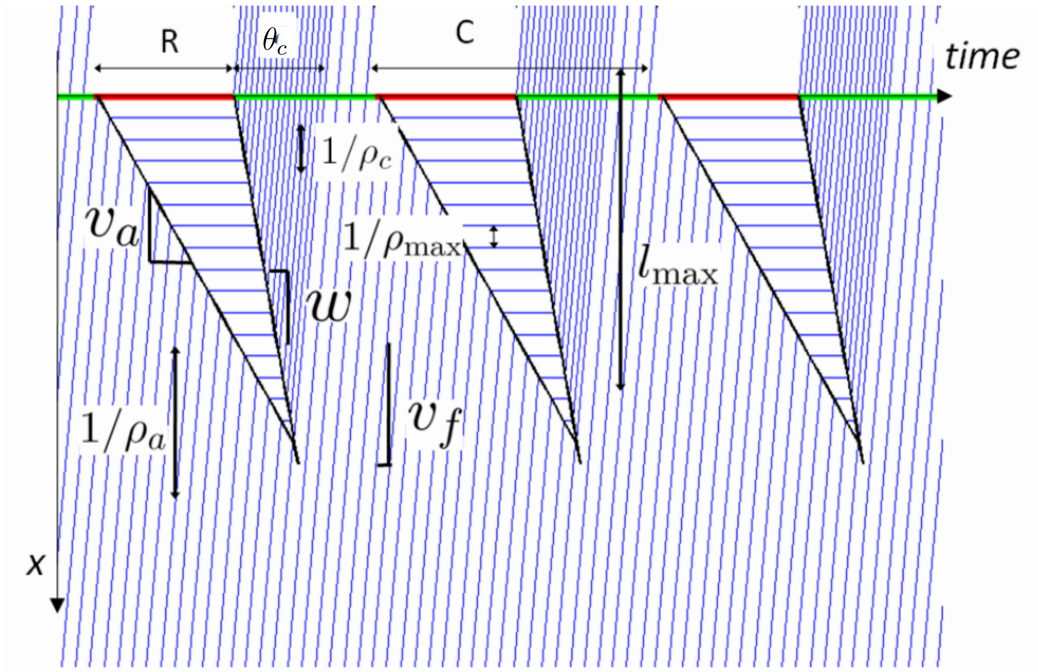


Figure 1.2: Space time diagram of vehicles trajectories under uniform arrivals of density ρ_a for an undersaturated regime

Chapter 2

Intersection seen as a system

2.1 Introduction to the model

We model intersections as a a system in which the input is the flow of vehicles arriving on a link and the output is the flow of vehicles leaving the link. Inputs and outputs are characterized by platoons of vehicles, defined by a density ρ_i and a duration T_i. Given the behaviour of the output of a single intersection in a general case, and the fact we want to study several intersections following each other, we choose to limit the structure of our input to three different platoons representing the red light duration, the queue discharge, and the extra green time of the previous intersection. As the extra green time does not have any constraints, it is not a platoon in general. However, to get a constant density, we average it. Although this seem an important approximation, it can be justified using Robertson's dispersion model [11] and the interaction between two platoons of different characteristics [5] making those two merging after some time. A balance equation allows to take the average to get the resulting density.

2.2 Notations and basic computations

Let's introduce some notations. ρ_c is the critical density introduced in the preliminaries and θ_c is the duration of the queue discharge we will compute explicitly later. R is the duration of the red light for this intersection. Given all these elements, we can compute the average density ρ_f of the last platoon using a balance equation :

$$\sum_{i=1}^{3} \rho_i T_i = \rho_c \theta_c + \rho_f (C - R - \theta_c)$$

$$\Longrightarrow \rho_f = \frac{\sum_{i=1}^{3} \rho_i T_i - \rho_c \theta_c}{C - R - \theta_c} \tag{2.1}$$

2.3 Characterization of the output platoons

Behaviour of a platoon through an intersection Each input platoon can only be confronted to four cases. Let's consider these cases for any platoon of density ρ_i and duration T_i.

Case 1. The entire platoon goes through the intersection during a green light period. The output platoon is of density ρ_i and of duration T_i.

Case 2. The beginning of the platoon goes through the intersection during a green light period but some vehicles at the end of the platoon stop at the red light. Let's notice that this case can only occur once in a cycle. The platoon is thus split in two parts, one of duration αT_i and another of duration $(1-\alpha)T_i$. The corresponding output is a platoon of density ρ_i and of duration αT_i followed by a platoon of density ρ_c and of duration $(1-\alpha)T_i\frac{\rho_i}{\rho_c}$ (obtained from a balance equation). This case is illustrated in the figure 2.1.

Case 3. All the vehicles of the platoon stop at the red light. The output platoon comes from the partial queue discharge and its duration is $T_i\frac{\rho_i}{\rho_c}$ (obtained from the same balance equation).

Case 4. The first vehicles of the platoon stop at the red light but the last ones go through the intersection without stopping. The output is the same as the one in Case 2, except that α represents the second part of the platoon, not the first. Again this case can only happen once in a cycle. This case is illustrated in the figure 2.2.

Let's consider the first car of the platoon i stops and waits during a time Δ. Then, the delay experienced by the following cars is growing linearly. Indeed, expressions derived from the Rankine-Hugoniot [3] jump conditions showed the speed of formation v_i and dissolution ω of the queue are constant for each platoon i :

$$v_i = \frac{\rho_i v_f}{\rho_{max} - \rho_i} \text{ and } \omega = \frac{\rho_c v_f}{\rho_{max} - \rho_c} \tag{2.2}$$

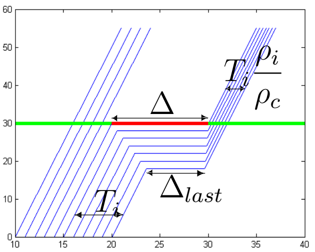


Figure 2.1: The platoon has a too small duration to generate a whole triangle.

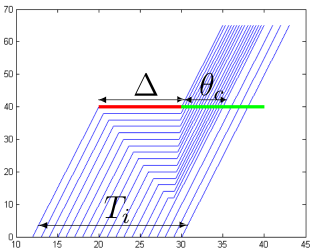


Figure 2.2: The platoon lasts enough time to generate a whole triangular queue.

Thus, the time Δ_{last} the last car of the platoon waits depends only on Δ and the characteristics of this platoon. We derive from a time balance equation illustrated in the figure 2.1:

$$\Delta_{last} = \Delta - T_i(1 - \frac{\rho_i}{\rho_c}) \tag{2.3}$$

However, this does not hold anymore if the last car of the platoon goes through the intersection without stopping. The previous case happens if and only if $T_i \leq \Delta + \theta_c \Leftrightarrow \Delta_{last} \leq 0$ as in the figure 2.2.

Computation of the output platoons in a general case After these primary results, let's consider a cycle starting with the red period and characterize the durations and densities of the output platoons given the durations and the densities of the input platoons. We denote by 1 the index of the platoon which hits the red light at the beginning of the cycle. Note that we can always choose a cyclic permutation on the indices of the input platoons and that this notation does not limit the generality of our results.

The first platoon is split as described in the second case with α depending on the offset between the beginning of the red cycle and the beginning of the first platoon. In a more general case, it corresponds to the offset between this traffic light and the previous one. Although the delay experienced is R , we will do the derivations with Δ for the sake of generality. So the output is either given by Case 2 or Case 4 corresponding to this first platoon. We sum up this case by introducing $\tau_1 = \max(0, \Delta\frac{\rho_c}{\rho_c - \rho_1})$, which is the duration of the fraction of the first platoon in which vehicles stop at the red light. Then the output is a first platoon of density ρ_c and of duration $\min(\alpha T_1, \tau_1)\frac{\rho_1}{\rho_c}$ followed by a second platoon of density ρ_2 and of duration $\max(0, T_1 - \tau_1)$.

The τs are introduces for the three input platoons but Δ changes every time according to the computation from last paragraph.

We summarize now the effect of such an intersection :

$$\left\{\begin{array}{c}(\rho_1, T_1)\\(\rho_2, T_2)\\(\rho_3, T_3)\end{array}\right\} \longmapsto \left\{\begin{array}{c}\Big(0, R\Big)\\ \Big(\rho_c, \min(\alpha T_1, \tau_1)\frac{\rho_1}{\rho_c}\Big)\\ \Big(\rho_1, \max(0, \alpha T_1 - \tau_1)\Big)\\ \Big(\rho_c, \min(T_2, \tau_2)\frac{\rho_2}{\rho_c}\Big)\\ \Big(\rho_2, \max(0, T_2 - \tau_2)\Big)\\ \Big(\rho_c, \min(T_3, \tau_3)\frac{\rho_3}{\rho_c}\Big)\\ \Big(\rho_3, \max(0, T_3 - \tau_3)\Big)\\ \Big(\rho_c, \min((1-\alpha)T_1, \tau_4)\frac{\rho_1}{\rho_c}\Big)\\ \Big(\rho_1, \max(0, (1-\alpha)T_1 - \tau_4)\Big)\end{array}\right\} \tag{2.4}$$

with

- $\tau_1 = R\dfrac{\rho_c}{\rho_c - \rho_1}$
- $\tau_2 = \max\Big(0, R - \alpha T_1(1 - \frac{\rho_1}{\rho_c})\Big)\dfrac{\rho_c}{\rho_c - \rho_2}$
- $\tau_3 = \max\Big(0, R - \alpha T_1(1 - \frac{\rho_1}{\rho_c}) - T_2(1 - \frac{\rho_2}{\rho_c})\Big)\dfrac{\rho_c}{\rho_c - \rho_3}$
- $\tau_4 = \max\Big(0, R - \alpha T_1(1 - \frac{\rho_1}{\rho_c}) - T_2(1 - \frac{\rho_2}{\rho_c}) - T_3(1 - \frac{\rho_3}{\rho_c})\Big)\dfrac{\rho_c}{\rho_c - \rho_1}$

For now, there is no contribution in the traffic from the side streets. Nevertheless, it will be included later with a density ρ_{cross} representing the incoming flow during the red light and the parameter ϵ representing the turning ratio during the green time.

Output platoons in an averaging intersection system This characterizes the system and thus we can average to get only three platoons as an output.

$$\left\{\begin{array}{c}(\rho_1, T_1)\\(\rho_2, T_2)\\(\rho_3, T_3)\end{array}\right\} \longmapsto \left\{\begin{array}{c}\Big(0, R\Big)\\ \Big(\rho_c, \theta_c\Big)\\ \Big(\rho_f, C - R - \theta_c\Big)\end{array}\right\} \tag{2.5}$$

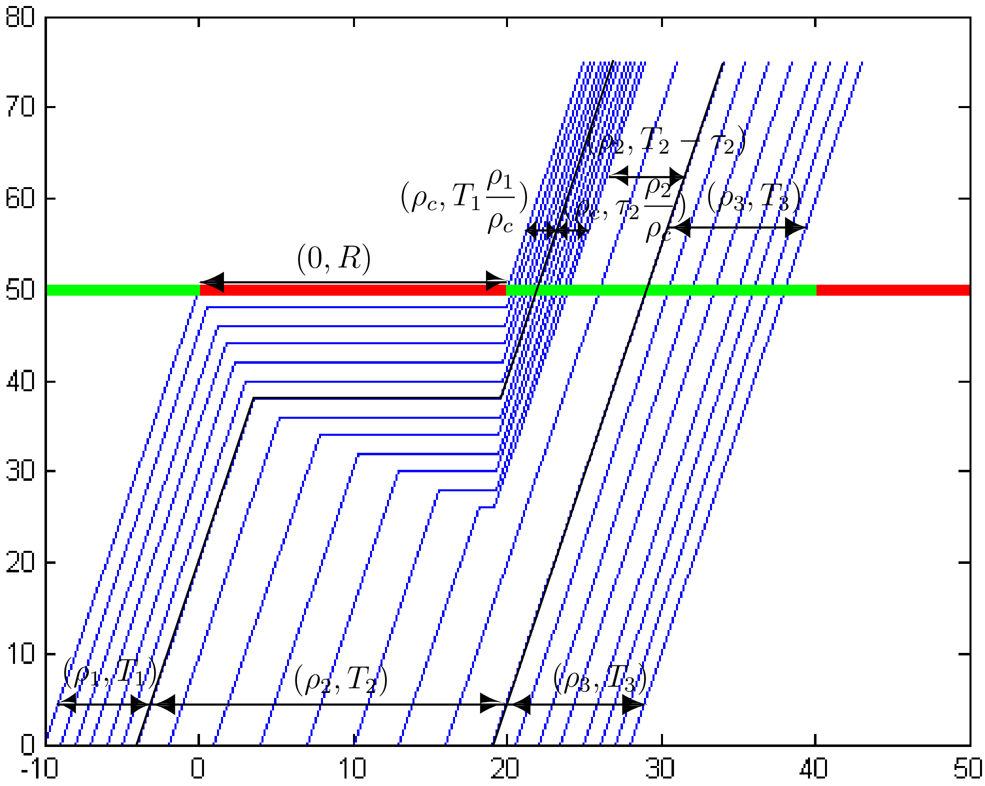


Figure 2.3: Example of output platoons in a space-time diagram referring to the formula (2.4)

with

- $\theta_c = \min(\alpha T_1, \tau_1)\frac{\rho_1}{\rho_c} + \min(T_2, \tau_2)\frac{\rho_2}{\rho_c} + \min(T_3, \tau_3)\frac{\rho_3}{\rho_c} + \min((1-\alpha)T_1, \tau_4)\frac{\rho_1}{\rho_c}$ the duration of the queue discharge
- ρ_f the merging density which only depends on θ_c and the parameters of the intersection as computed in (2.1)

Chapter 3

Optimization of a two traffic lights system

3.1 Computation of the cost function

Choice of the cost function We want to optimize the vehicle flow through two intersections. We decided to choose as a cost function the total delay Γ experienced at the second red light by all the vehicles going through the intersections.

$$\Gamma = \sum_{vehicles} W_k \tag{3.1}$$

where W_k is the waiting time experienced by the car k at the second red light.

We take such a cost function because it appears to be the most fair from a social point of view, in the sense we do not choose to make some vehicles wait a long time in order to make some vehicles going through without stopping. Other choices are possible such as maximize the number of vehicles going through without stopping or minimize the maximal delay experienced but did not seem as socially fair as the one we chose.

Let's start with computing this cost function, using the previous results and assuming that the platoons are coming from an upstream intersection, giving the input this particular three-platoons structure.

Computation for a single platoon As shown in (2.2), the delay is growing linearly inside a platoon. So, the total delay for the platoon only depends on the delays experienced by the first car and the last car. These delays correspond to the actual Δ and Δ_{last} mentioned in (2.3). We thus define the Δ_i coming from the expressions of the τ_i and illustrated by the figure 3.1:

- $\Delta_1 = R$
- $\Delta_2 = \max\left(0, R - \alpha T_1(1 - \frac{\rho_1}{\rho_c})\right)$
- $\Delta_3 = \max\left(0, R - \alpha T_1(1 - \frac{\rho_1}{\rho_c}) - T_2(1 - \frac{\rho_2}{\rho_c})\right)$
- $\Delta_4 = \max\left(0, R - \alpha T_1(1 - \frac{\rho_1}{\rho_c}) - T_2(1 - \frac{\rho_2}{\rho_c}) - T_3(1 - \frac{\rho_3}{\rho_c})\right)$
- $\Delta_5 = \max\left(0, R - T_1(1 - \frac{\rho_1}{\rho_c}) - T_2(1 - \frac{\rho_2}{\rho_c}) - T_3(1 - \frac{\rho_3}{\rho_c})\right)$

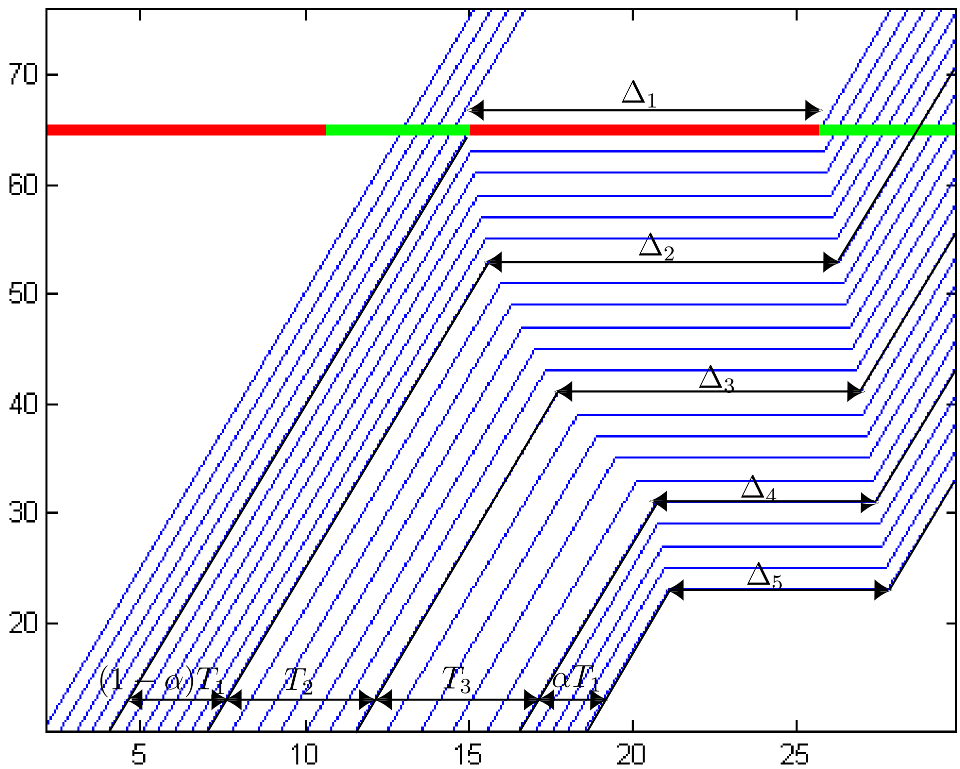


Figure 3.1: Waiting times of the first and last cars of each platoon

The total delay experienced by the vehicles of a platoon is the average delay of the stopping vehicles times the number of stopping vehicles. The duration of the platoon containing stopping vehicles is either the duration T_i of the platoon or the duration τ_i of the part of the platoon which is actually experiencing some delay.

Dependence on the control variables The only parameters we control in order to get our optimization is the duration of the red light and the offset between the two traffic lights. However, it is not relevant to minimize according to the duration of the red light because, without any constraints, the optimal value is null. We can only control the actual offset Θ between the two traffic lights. We introduce the standardized offset $t_0 = \Theta - \frac{L}{v_f}$, which takes into account the time vehicles take to go from the upstream intersection to the downstream one. Here, L represents the length of the link between the two intersections.

The standardized offset can be introduced in the expression of the cost function (3.2) by noticing that $t_0 = (1-\alpha)T_1$, which gives us the explicit dependence of the total delay on t_0. The implicit one is the cyclic permutation between the platoons depending on which interval the offset belongs to.

Expression of the total delay Then, assuming all the hypothesis made previously, we derive the expression of the total delay Γ :

$$\begin{aligned}\Gamma = \rho_1 \min(\tau_1, T_1 - t_0)\frac{\Delta_1 + \Delta_2}{2} + \rho_2 \min(\tau_2, T_2)\frac{\Delta_2 + \Delta_3}{2} \\ + \rho_3 \min(\tau_3, T_3)\frac{\Delta_3 + \Delta_4}{2} + \rho_1 \min(\tau_4, t_0)\frac{\Delta_4 + \Delta_5}{2} \quad (3.2)\end{aligned}$$

Let's notice that this expression still holds in the case of a congested regime.

Remark. *The real total delay would be in fact* $v_f\Gamma$, *where* v_f *is the free flow velocity, but as it is a constant, minimizing the total delay is equivalent to minimizing* Γ.

3.2 Study of the convexity of the cost function

Particular structure of the cost function For this paragraph, we take the following convention, $\widetilde{T_1} = \alpha T_1 = T_1 - t_0$, $\widetilde{T_4} = (1-\alpha)T_1 = t_0$ and $\rho_4 = \rho_1$ to simplify the notations in the equations. Not to get confused, we denote also T_2 and T_3 respectively by $\widetilde{T_2}$ and $\widetilde{T_3}$.

Moreover, we have $\forall i$, $\tau_i \leqslant \tau_{i-1}$ i.e. the $\tau_i s$ are decreasing. Especially, once one is null, the followings are null too. The same results apply for the Δs because $\tau_i = \dfrac{\rho_c}{\rho_c - \rho_i}\Delta_i$, which means that τ_i ans Δ_i are simultaneously equal to 0.

Now, we derive some properties of the cost function.

Property 1. *In an undersaturated regime,* $\forall t_0$, $\exists! k \in \{1,\dots,4\}$ *such that* $0 < \tau_k \leqslant \widetilde{T}_k$.

We thus derive a new expression for the cost function :

$$\Gamma = \sum_{i=1}^{k-1} \rho_i \widetilde{T}_i \frac{\Delta_i + \Delta_{i+1}}{2} + \rho_k \frac{\rho_c}{\rho_c - \rho_k} \frac{\Delta_k^2}{2} \tag{3.3}$$

Proof. Let's assume such a k exists, then :

$$\begin{aligned}
\tau_k > 0 \iff & R - \sum_{i=1}^{k-1} \widetilde{T}_i (1 - \frac{\rho_i}{\rho_c}) > 0 \\
\iff & R - \sum_{i=1}^{k-2} \widetilde{T}_i (1 - \frac{\rho_i}{\rho_c}) > \widetilde{T}_{k-1} (1 - \frac{\rho_{k-1}}{\rho_c}) \\
\iff & \tau_{k-1} > \widetilde{T}_{k-1}
\end{aligned}$$

This implies $\tau_{k-1} > 0$ and we derive by induction that $\forall j < k, \tau_j > T_j$.

We also have :

$$\begin{aligned}
\tau_k \leqslant \widetilde{T}_k \iff & \frac{\rho_c}{\rho_c - \rho_k} \left(R - \sum_{i=1}^{k-1} \widetilde{T}_i (1 - \frac{\rho_i}{\rho_c}) \right) \leqslant \widetilde{T}_k \\
\iff & R - \sum_{i=1}^{k-1} \widetilde{T}_i (1 - \frac{\rho_i}{\rho_c}) \leqslant \widetilde{T}_k (1 - \frac{\rho_k}{\rho_c}) \\
\iff & R - \sum_{i=1}^{k} \widetilde{T}_i (1 - \frac{\rho_i}{\rho_c}) \leqslant 0 \\
\iff & \tau_{k+1} \leqslant 0
\end{aligned}$$

This implies $\tau_{k+1} \leqslant \widetilde{T}_{k+1}$ and we derive by induction that $\forall j > k, \tau_j \leqslant 0$.

This shows that if such a k exists, it is unique. We will demonstrate now its existence.

Let $j = \max\{q \in \{1,\dots,4\} | \tau_q > \widetilde{T}_q\}$. This set is bounded because, if the regime is undersaturated, $\tau_4 \leqslant t_0$, so j exists. As shown earlier, $\tau_j > \widetilde{T}_j \Rightarrow \tau_{j+1} > 0$ and $\tau_{j+1} \leqslant \widetilde{T}_{j+1}$, otherwise j is not the maximum of the set. This shows the existence of this index k. □

Remark. *The index k depends on t_0 but is piecewise constant. So if we study the cost function over the right interval, this expression of Γ holds.*

Physically, k represents the index of the first platoon in which some vehicles go through the intersection without stopping. Moreover, the expression holds in the case of a congested regime, with $k = 5$ and the convention $\rho_5 = 0$.

Property 2. Γ *is piecewise quadratic with respect to* t_0.

Proof. We study the cost function over an interval where k is constant. We thus use the expression computed in the property (1). The variable t_0 appears there in the Δ_i linearly and in $\widetilde{T_1}$ linearly too. $\widetilde{T_4}$ never appears in the equation.

So, all the terms of the sum from 2 to $k-1$ are linear in t_0. The first term of the sum is in t_0^2, so is the last term of the expression. Therefore, Γ can be rewritten as $at_0^2 + bt_0 + c$ on each interval where k is constant.

After computation, we get :

$$a = \frac{(\rho_c - \rho_1)(\rho_k - \rho_1)}{2(\rho_c - \rho_k)} \tag{3.4}$$

$$b = -\frac{R\rho_c(\rho_1 - \rho_k) - \sum_{i=1}^{k-1} T_i(\rho_c - \rho_1)(\rho_i - \rho_k)}{\rho_c - \rho_k} \tag{3.5}$$

and the optimum (either a minimum or a maximum according to the sign of a) is reached in :

$$-\frac{b}{2a} = \sum_{i=1}^{k-1} T_i \frac{\rho_k - \rho_i}{\rho_k - \rho_1} - R\frac{\rho_c}{\rho_c - \rho_1} \tag{3.6}$$

□

Remark. *In these equations, T_1 is now referring to the real duration of the first platoon and this is why we don't use $\widetilde{T_1}$ anymore.*

Study of the variations of the cost function Since Γ is piecewise quadratic, we study its monotony on each interval where k is constant in order to determine where the global optimum is. Such a study leads to the following property.

Property 3. *If we choose* 0 *as the beginning of the platoon with the highest density,* Γ *is a quasi-convex function.*

Proof. We study the monotony of Γ over each interval corresponding to a platoon - i.e. over $[0, T_1]$, $[T_1, T_1 + T_2]$, $[T_1 + T_2, C]$. In this case, the densities ρ_i are fixed, but k is not. Nevertheless, it can only increase. We study the variations of Γ to prove the global shape of the function is decreasing then increasing with sometimes an interval over which the function is constant.

- First interval : $\rho_1 = \max_i \rho_i$
  - $k = 1$: In this case, we have the coefficients a and b that are null, Γ is thus constant.
  - $k = 2$: We have
$$-\frac{b}{2a} = T_1 - R\frac{\rho_c}{\rho_c - \rho_1}$$
    and we know that
$$\begin{aligned} \tau_2 > 0 &\iff R - (T_1 - t_0)(1 - \frac{\rho_1}{\rho_c}) > 0 \\ &\iff t_0 > T_1 - R\frac{\rho_c}{\rho_c - \rho_1} \\ &\iff t_0 > -\frac{b}{2a} \end{aligned}$$
    Thus we only have the right side of the parabola. This side may be increasing or decreasing according to the sign of a. We can derive from (3.4) that a has the same sign as $(\rho_k - \rho_1)$, here nonpositive because ρ_1 is the highest density, so Γ is decreasing.
  - $k = 3$: We have
$$-\frac{b}{2a} = T_1 + T_2\frac{\rho_3 - \rho_2}{\rho_3 - \rho_1} - R\frac{\rho_c}{\rho_c - \rho_1}$$
    which is not related to the condition $\tau_3 > 0$. However, we derive from this condition a lower bound for t_0 :
$$\begin{aligned} \tau_3 > 0 &\iff R - T_1(1 - \frac{\rho_1}{\rho_c}) - T_2(1 - \frac{\rho_2}{\rho_c}) + t_0(1 - \frac{\rho_1}{\rho_c}) > 0 \\ &\iff t_0 > T_1 + T_2\frac{\rho_c - \rho_2}{\rho_c - \rho_1} - R\frac{\rho_c}{\rho_c - \rho_1} \end{aligned}$$
    This lower bound t_{min} is less than $-\frac{b}{2a}$ and thus we still have only one side of the parabola.
$$\begin{aligned} t_{min} + \frac{b}{2a} &= T_2\frac{\rho_c - \rho_2}{\rho_c - \rho_1} - T_2\frac{\rho_3 - \rho_2}{\rho_3 - \rho_1} \\ &= T_2\frac{(\rho_c - \rho_3)(\rho_2 - \rho_1)}{(\rho_c - \rho_1)(\rho_3 - \rho_1)} \\ &\geqslant 0 \text{ as } \rho_1 \text{ is the highest density} \end{aligned}$$

So we actually still have the right side of the parabola, and a is still nonpositive, meaning Γ is decreasing.

  - $k = 4$: In this case, as $\rho_k = \rho_1$, we have $a = 0$ and $b = \sum_{i=1}^{3} T_i(\rho_i - \rho_1)$ which is negative as ρ_1 is the highest density. So Γ is decreasing.

In any case, if $\rho_1 = \rho_c$, we have $a = 0$ and Γ is linear with a negative slope $-R\rho_c$.

To conclude, Γ is nonincreasing over the interval $[0; T_1]$.

- Second interval : $\rho_1 = \min_i \rho_i$

  - $k = 1$: Idem as the first interval, Γ is constant.
  - $k = 2$: For the same reason as previously, we only have the right side of the parabola. Nevertheless, this time a is nonnegative as ρ_1 is the lowest density, so Γ is increasing.
  - $k = 3$: Although ρ_1 is now the lowest density, the condition $t_{min} + \dfrac{b}{2a} \geqslant 0$ holds because the sign of two factors have changed at the same time and the other factors keep their sign. So we still have the right side of the parabola and Γ is increasing.
  - $k = 4$: For the same reason as previously, Γ is linear and the sign of the slope is positive, meaning Γ is increasing.

Here we can't have $\rho_1 = \rho_c$, else all the ρ_i would be equal to ρ_c and the queue could never be discharged.

To conclude, Γ is increasing over the interval corresponding to ρ_1 as the lowest density.

- Last interval : ρ_1 is neither the highest nor the lowest density

  - $k = 1$: Γ is still constant in this case.
  - $k = 2$: We still have the right side of the parabola. The sign of a is the sign of $(\rho_2 - \rho_1)$ which is the same sign as $(\rho_2 - \rho_3)$ because ρ_1 is the intermediate density. Moreover $(\rho_2 - \rho_3)$ is the sign of a for the previous interval (because of the cyclic permutation we applied when we changed intervals), so Γ will have the same monotony as it has over the previous interval. The previous interval corresponds to the platoon indexed 3 for the study of this interval.

- $k = 3$: Here, $t_{min} + \dfrac{b}{2a} \leqslant 0$ as ρ_1 is the intermediate density. So we do have the optimum in the parabola unless this optimum is reached after the upper bound of t_0. The upper bound of t_0 is given by :

$$\begin{aligned}\tau_3 \leqslant T_3 &\iff R - T_1(1-\frac{\rho_1}{\rho_c}) - T_2(1-\frac{\rho_2}{\rho_c}) + t_0(1-\frac{\rho_1}{\rho_c}) \leqslant T_3(1-\frac{\rho_3}{\rho_c}) \\ &\iff t_0 \leqslant T_1 + T_2\frac{\rho_c-\rho_2}{\rho_c-\rho_1} + T_3\frac{\rho_c-\rho_3}{\rho_c-\rho_1} - R\frac{\rho_c}{\rho_c-\rho_1}\end{aligned}$$

Let's compare this upper bound t_{max} with where the optimum is reached.

$$\begin{aligned}-\frac{b}{2a} - t_{max} &= -\frac{b}{2a} - t_{min} - T_3\frac{\rho_c-\rho_3}{\rho_c-\rho_1} \\ &= T_2\frac{(\rho_c-\rho_3)(\rho_1-\rho_2)}{(\rho_c-\rho_1)(\rho_3-\rho_1)} - T_3\frac{\rho_c-\rho_3}{\rho_c-\rho_1} \\ &= \frac{\rho_c-\rho_3}{\rho_c-\rho_1}\left(\frac{T_2(\rho_1-\rho_2) - T_3(\rho_3-\rho_1)}{\rho_3-\rho_1}\right) \\ &= \frac{\rho_c-\rho_3}{\rho_c-\rho_1}\left(\frac{C\rho_1 - \sum_{i=1}^3 \rho_i T_i}{\rho_3-\rho_1}\right)\end{aligned}$$

We derive two conditions from this expression for the optimum of the parabola not to be reached.

$$\text{if} \begin{cases} \rho_3 > \rho_1 \\ \rho_1 > \dfrac{1}{C}\sum_{i=1}^3 \rho_i T_i \end{cases} \quad \text{or} \quad \begin{cases} \rho_3 < \rho_1 \\ \rho_1 < \dfrac{1}{C}\sum_{i=1}^3 \rho_i T_i \end{cases} \tag{3.7}$$

We have the left side of the parabola and a has the sign of $(\rho_3 - \rho_1)$. If those two conditions are not fulfilled, we have the optimum of the parabola.

If $\rho_3 > \rho_1$, we have a minimum, but it also means that ρ_3 is the highest value and thus, over the interval corresponding to ρ_3 (in fact the one prior to the one we are studying), Γ is decreasing. So, there is still one minimum.

Else, we have a maximum, but ρ_3 is now the lowest value and thus Γ was increasing over the previous interval.

- $k = 4$: Γ is linear with a slope of same sign as $\dfrac{1}{C}\sum_{i=1}^3 \rho_i T_i - \rho_1$.

To conclude, if Γ is monotonous over this interval, we have a point c where Γ is nonincreasing $\forall t_0 \leqslant c$ and nondecreasing $\forall t_0 \geqslant c$. Given the results of the case $k = 3$, this holds when Γ is not monotonous over this interval. We can even notice that, if k is increasing over this interval, the condition still holds. Only one exception appears, when we got the maximum of the parabola over the interval, we have to do the right translation (as Γ is periodic) to make this hold.

Thus, if we choose the value where $t_0 = 0$ as the beginning of the platoon with the highest density or as the maximum of the parabola when this one appears, Γ is a quasi-convex function over the interval [0;C]. □

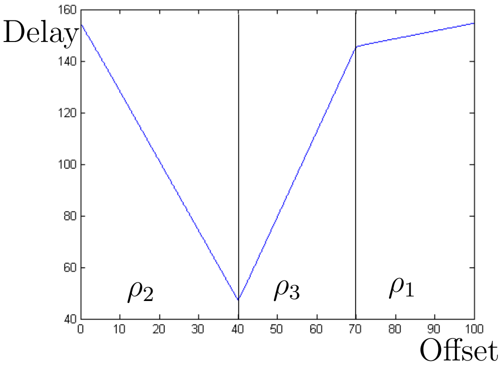
3.3 Identification of the different scenarios

As we know the variations of Γ over $[0; C]$, we are able to find analytically the optimal control to apply. We notice two families of solutions depending on the parameters, the corner solutions where t_0 is at the beginning of a platoon and the solutions inside an interval. These solutions are only in the interval corresponding to the intermediate density, according to the previous results. From now, we will index this intermediate density by 1. The optimal t_0 will be noted t_0^*.

Then, we want to study several consecutive intersections. As optimizing the whole sum of all the cost functions for each intersection is a hard problem, we try to optimize each intersection at a time and see, given the output platoons after applying the optimal control, which control is to be applied at the downstream intersection. A scenario is unstable if the scenario reached for the next intersection is different from the initial one. In other terms, a scenario is considered unstable when the solution jumps from a scenario to another after an intersection. On the contrary, a scenario is stationary if we apply the same optimal control for all the following downstream intersections.

We will try in this paragraph to identify the conditions for each solution to occur and to see if these conditions hold for the next intersection, making this scenario stationary.

The solution $t_0^* \notin [0; T_1]$: This solution occurs if and only if $\rho_2 \geqslant \rho_1 \geqslant \rho_3$, and then $t_0^* = T_1 + T_2$. This solution is considered as a corner solution. It means a platoon is chosen such that its first car will hit the traffic light exactly when it starts its red time. The optimization shows that this platoon is the one with the lowest density. This result is intuitive because the vehicles who stop first will wait the longest, so, to minimize the delay, they should be in a platoon where vehicles are far away from each other. Furthermore, in this case, the platoon with the highest density is at the end and will more likely not stop at the intersection or, if some vehicles stop, they will not wait very long.



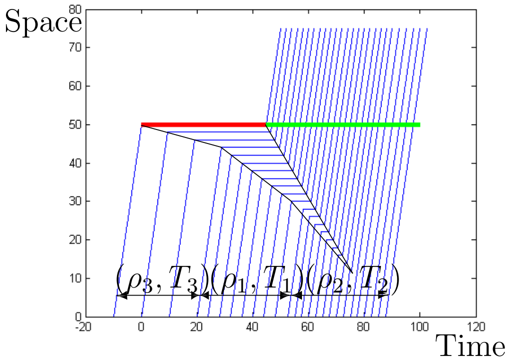


Figure 3.2: Example of a case in which $t_0^* \notin [0; T_1]$ with : $\rho_1 = 0.05, \rho_2 = 0.08, \rho_3 = 0.02, T_1 = 30, T_2 = 40, T_3 = 30, C = 100, R = 45, \rho_c = 0.1$

We can see the output platoons are $(\rho_{cross}, R), (\rho_c(1-\epsilon), \theta_c), (\rho_f(1-\epsilon), C-R-\theta_c)$, with ϵ some noise expected to model the ratio of vehicles turning at this intersection. If we consider this road as a main street, we have $\epsilon \ll 1$ and $\rho_{cross} \ll \bar{\rho}$ where $\bar{\rho} = \frac{1}{C}\sum_{k=1}^{3} \rho_i T_i$ is the average density of the input traffic flows. We derive the new conditions on the input for the downstream intersection : $\rho_3 \geqslant \rho_1 \geqslant \rho_2$. The initial conditions are not fulfilled anymore, making this scenario unstable.

The solution $t_0^* \in]0; T_1[$: This is the only case in which the solution is not a corner solution. The first platoon, with the intermediate density, is split in two parts as illustrated in the figure 3.3. This seems optimal because if more vehicles from the first platoon wait at the red light (offset increasing), it will not be compensated by the important number of vehicles from the third platoon - with the highest density - which had to wait only a short time. On the other hand, if less vehicles from the first platoon wait at the red light (offset decreasing), it will be overcompensated by a lot of vehicles from the third platoon - with the highest density - which would have to wait longer. This is thus a trade-off between a few cars which wait long and a lot of cars which wait a short time.

This solution occurs if and only if the regime is undersaturated and

$$\begin{cases} \rho_2 \leqslant \rho_1 \leqslant \bar{\rho} \leqslant \rho_3 \\ T_2 \leqslant R\dfrac{\rho_c}{\rho_c - \rho_2} \\ \dfrac{R}{C} \leqslant \dfrac{(\rho_3 - \bar{\rho})(\rho_c - \rho_1)}{\rho_c(\rho_3 - \rho_1)} \end{cases} \tag{3.8}$$

Then $t_0^* = T_1 + T_2\dfrac{\rho_3 - \rho_2}{\rho_3 - \rho_1} - R\dfrac{\rho_c}{\rho_c - \rho_1}$

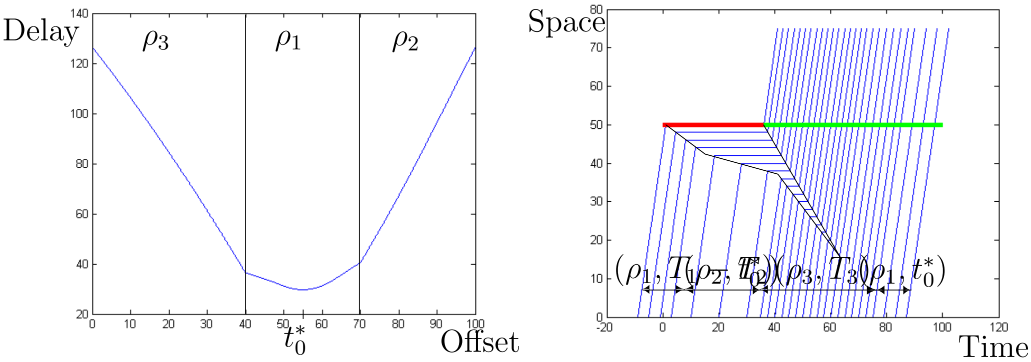


Figure 3.3: Example of a case in which $t_0^* \in]0; T_1[$ with : $\rho_1 = 0.04, \rho_2 = 0.02, \rho_3 = 0.08, T_1 = 30, T_2 = 30, T_3 = 40, C = 100, R = 36, \rho_c = 0.1$

After computation, we have the following output platoons :

$$(\rho_{cross}, R), (\rho_c(1-\epsilon), R\frac{\rho_1}{\rho_c - \rho_1}), (\rho_f(1-\epsilon), C - R\frac{\rho_c}{\rho_c - \rho_1})$$

where

$$\begin{aligned} \rho_f &= \frac{\bar{\rho}C - \rho_1 R\frac{\rho_c}{\rho_c - \rho_1}}{C - R\frac{\rho_c}{\rho_c - \rho_1}} \\ &= \bar{\rho} + \frac{(\bar{\rho} - \rho_1)R\frac{\rho_c}{\rho_c - \rho_1}}{C - R\frac{\rho_c}{\rho_c - \rho_1}} \\ &\geqslant \bar{\rho} \end{aligned}$$

Thus, with the hypothesis of this road as a main street, we can see that the condition $\rho_3 \geqslant \rho_1 \geqslant \rho_2$ (i.e. $\rho_c(1-\epsilon) \geqslant \rho_f(1-\epsilon) \geqslant \rho_{cross}$)holds. As the perturbations are very small, we consider at the first order that the number of vehicles is conserved, meaning the $\bar{\rho}$ from the downstream intersection is equal to one from the upstream intersection. So, as ρ_f is the new intermediate density, we do not have $\rho_1 \leqslant \bar{\rho}$ anymore. This scenario is also unstable.

The solution is $t_0^* = 0$: This solution is a corner solution too. The platoon with the highest density is in the last position not to make a lot of vehicles wait very long. However, the first platoon, which has the intermediate density must not have a too high density - it must remain under the average density $\bar{\rho}$ - because its vehicles wait the longest. If the density of the first platoon is too important, it is not profitable to make its vehicles waiting a long time.

This solution occurs if and only if

$$\begin{cases} \text{The regime is congested} \\ \rho_2 \leqslant \rho_1 \leqslant \bar{\rho} \leqslant \rho_3 \end{cases} \quad \text{or} \quad \begin{cases} \text{The regime is undersaturated} \\ \rho_2 \leqslant \rho_1 \leqslant \bar{\rho} \leqslant \rho_3 \\ T_2 \leqslant R\dfrac{\rho_c}{\rho_c - \rho_2} \\ \dfrac{R}{C} \geqslant \dfrac{(\rho_3 - \bar{\rho})(\rho_c - \rho_1)}{\rho_c(\rho_3 - \rho_1)} \end{cases}$$

As an output of a congested regime, we only have two platoons in the case of the figure 3.4, it is thus hard to determine at which scenario they belong, as the density ρ_1 of the platoon of null duration can have any value, because it is then multiplied by 0 in the equations. It is impossible to say if ρ_1 is the highest, lowest or intermediate density. Nevertheless, if we consider it as the intermediate density, what we implicitly did by calling it ρ_1, the solution of the downstream intersection is either 0 or T_1 depending on which is higher between $\bar{\rho}$ and ρ_1. It actually does not matter because $T_1 = 0$. For reasons we will see later, let's say it would switch to the scenario with the solution T_1.

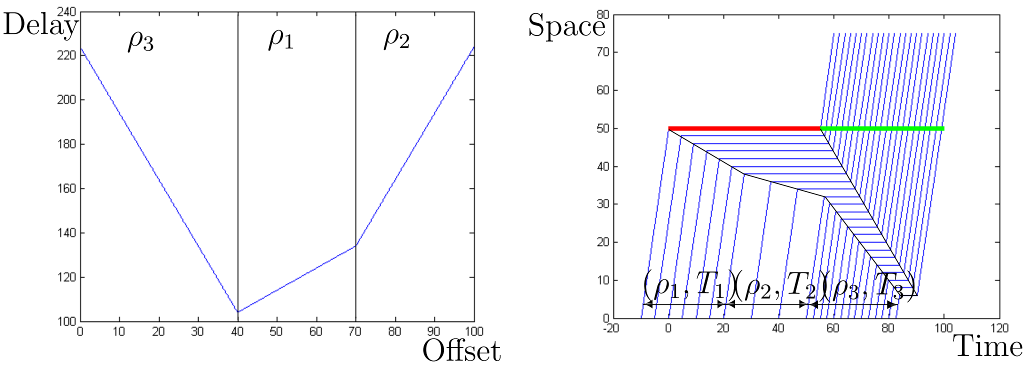


Figure 3.4: Example of a case in which $t_0^* = 0$ with : $\rho_1 = 0.04, \rho_2 = 0.02, \rho_3 = 0.08, T_1 = 30, T_2 = 30, T_3 = 40, C = 100, R = 55, \rho_c = 0.1$

In the case of the figure 3.5, the only platoon going through the intersection during the extra green time is the third one and thus $\rho_f = \rho_3$. With the hypothesis of this road as a main street, ρ_f should be considered as the intermediate density for the next intersection and $\rho_3 \geqslant \bar{\rho}$ according to the initials conditions and the conservation of the number of vehicles (at the first order of the approximation).We conclude this case is unstable too.

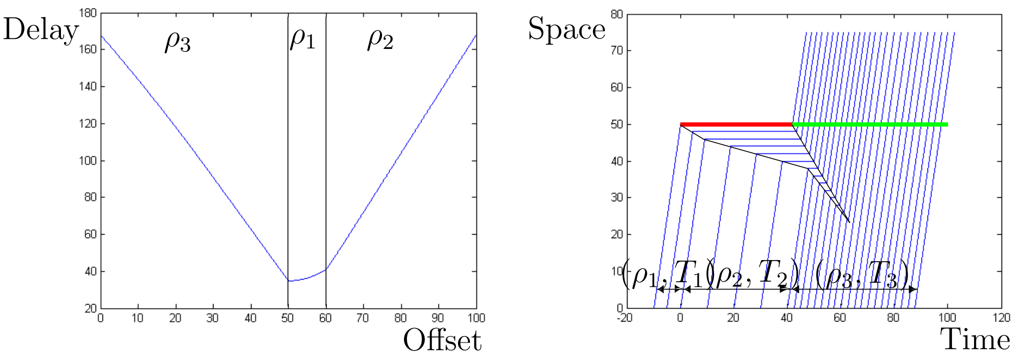


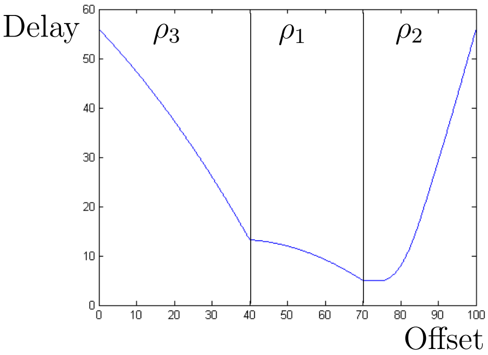
Figure 3.5: Example of a case in which $t_0^* = 0$ with : $\rho_1 = 0.04, \rho_2 = 0.02, \rho_3 = 0.08, T_1 = 10, T_2 = 40, T_3 = 50, C = 100, R = 42, \rho_c = 0.1$

The solution is T_1 : This corner solution occurs because the second condition expects the second platoon to go through the intersection without stopping. The best choice is thus to make the platoon with the lowest density wait at the red light. It does not have any consequences on the other platoons.

This solution occurs if and only if

$$\begin{cases} \rho_3 \geqslant \bar{\rho} \geqslant \rho_1 \geqslant \rho_2 \\ T_2 \geqslant R \dfrac{\rho_c}{\rho_c - \rho - 2} \end{cases} \quad \text{or } \rho_3 \geqslant \rho_1 \geqslant \bar{\rho} \geqslant \rho_2$$

In the case of the figure 3.6, the only platoon hitting the red light is the second one. So, we compute ρ_f :



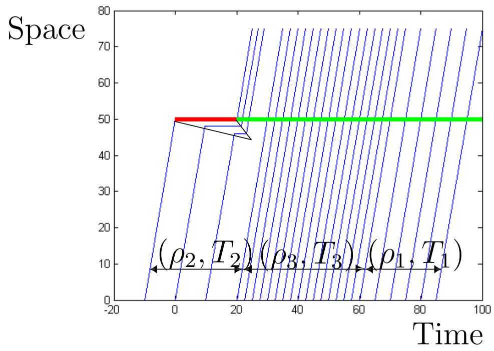


Figure 3.6: Example of a case in which $t_0^* = T_1$ with : $\rho_1 = 0.04, \rho_2 = 0.02, \rho_3 = 0.08, T_1 = 30, T_2 = 30, T_3 = 40, C = 100, R = 20, \rho_c = 0.1$

$$\begin{aligned}
\rho_f &= \frac{\rho_2(T_2 - \tau_2) + \rho_3 T_3 + \rho_1 T_1}{C - R\frac{\rho_c}{\rho_c - \rho_2}} \\
&= \frac{\bar{\rho} C - \rho_2 R\frac{\rho_c}{\rho_c - \rho_2}}{C - R\frac{\rho_c}{\rho_c - \rho_2}} \\
&= \bar{\rho} + \frac{(\bar{\rho} - \rho_2)\frac{\rho_c}{\rho_c - \rho_2}}{C - R\frac{\rho_c}{\rho_c - \rho_2}} \\
&\geqslant \bar{\rho}
\end{aligned}$$

As ρ_f is the intermediate density for the next intersection, the condition $\rho_1 \leqslant \bar{\rho}$ does not hold and we change the scenario.

In the case of the figure 3.7, we check all the cases depending on the value of k at the optimum. After computing all the values of ρ_f for these different cases, we still have the condition $\rho_3 \geqslant \rho_1 \geqslant \bar{\rho} \geqslant \rho_2$ for the next intersection. So, this scenario is stationary.

Link between the scenarios Now that we have identified all the possible scenarios, we study the interactions between them and the transitions from one to another.

The green arrows on the figure 3.3 mean that this is not a compulsory path, but the real path taken depends on several parameters. Nevertheless, we can see some kind of attractor in the left bottom of the figure, where all the scenarios converge after a finite number of iterations.

Physically, this attractor corresponds to what is called a green wave[4]. A green wave is a flow of vehicles going through a series of intersections without

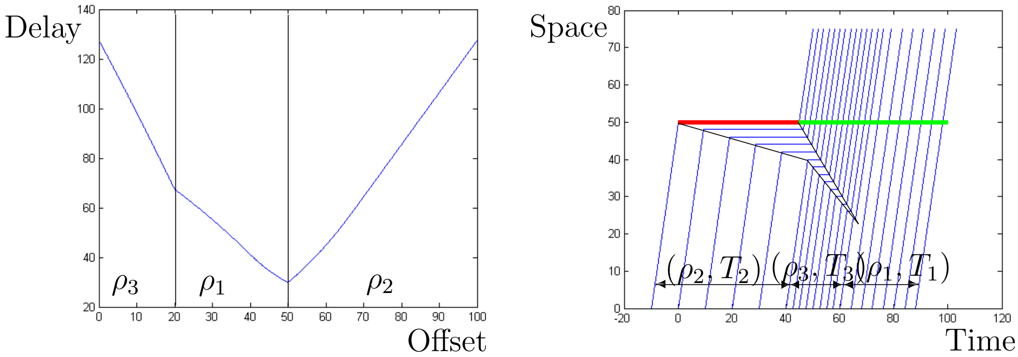


Figure 3.7: Example of a case in which $t_0^* = T_1$ with : $\rho_1 = 0.05, \rho_2 = 0.02, \rho_3 = 0.08, T_1 = 30, T_2 = 50, T_3 = 20, C = 100, R = 45, \rho_c = 0.1$

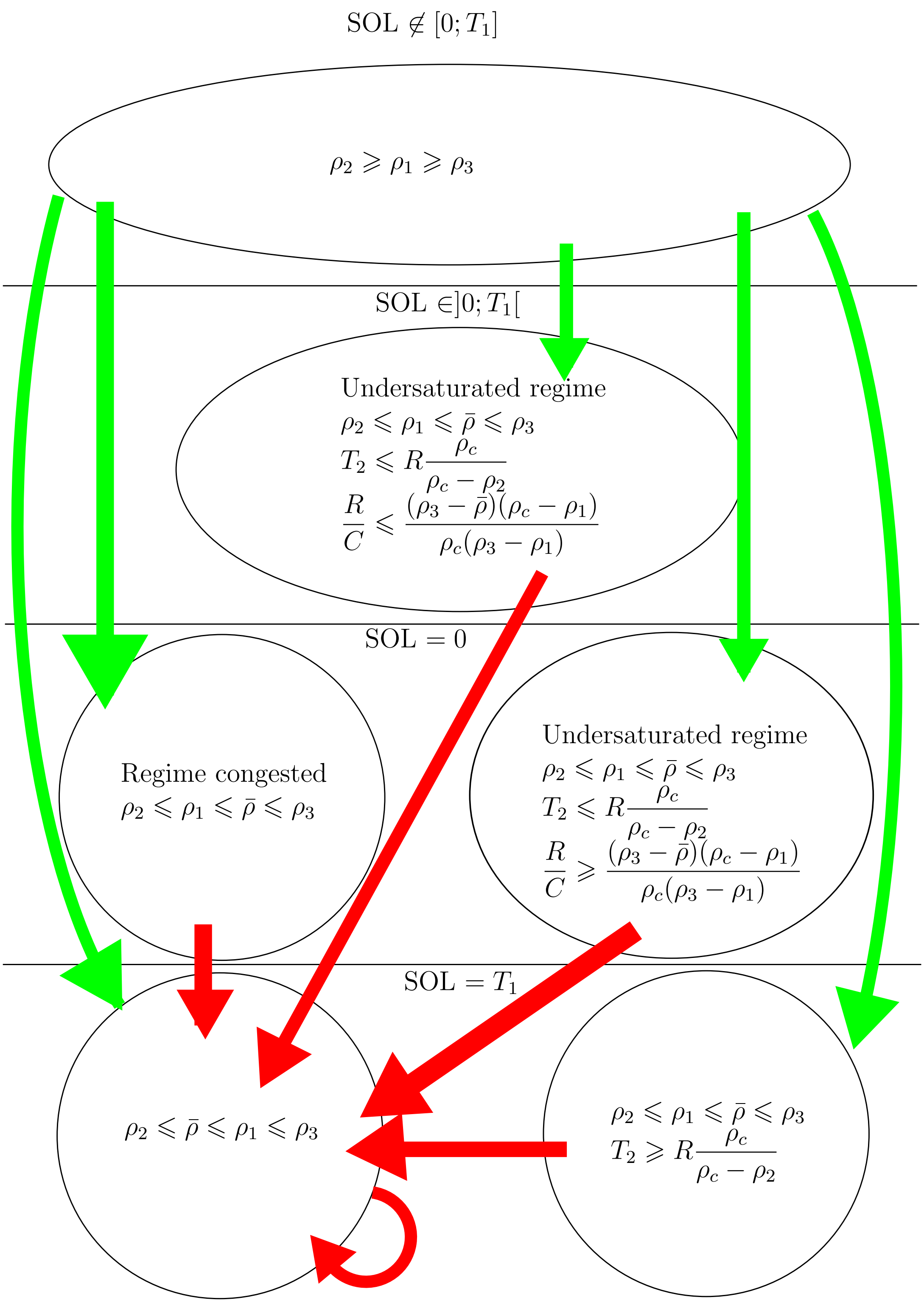
stopping at any red light. Here, we are converging to a green wave because, at each intersection, we have as an output a platoon with few vehicles, corresponding to the red light, and thus as the number of vehicles is conserved, two other platoons with a density increasing after each traffic light crossed, until it reached the critical density ρ_c.

In such a green wave, the platoon is so tight that it goes through the intersection during the green time without any vehicle stopping. This is possible as long as the regime is undersaturated, meaning the duration of the single platoon at critical density must be less than the green time of the intersection. This minimum, expected to be local because we only optimize each intersection at a time and not the entire set of intersections at once, is actually a global minimum because the cost function is null, it is not possible to do better. If the regime is congested, it is still optimal to do a green wave from a local point of view, but it is not sure if we optimize globally.

However, this green wave is not the ideal solution for synchronizing traffic lights because it is very sensitive to external conditions. Indeed, vehicles are really close to one another, so one small incident on the road, such as a pedestrian crossing the road while he was not supposed to, will cause a huge amount of delay reflecting on all the vehicles behind.

To improve this situation, we can choose to apply this optimal control in real-time. If we were able to know the traffic conditions at the downstream intersection with sensors for instance, we could decide to apply the right control and thus anticipate an incident which would have disrupted the green wave. This idea of real-time control traffic has already been studied with real-time computations [6, 9, 12]. Here, the choice is decided with comparisons between parameters, the time of computation is thus really short.

Although this model does not hold when the traffic from the side streets



SOL $\notin [0;T_1]$
$\rho_2 \geqslant \rho_1 \geqslant \rho_3$
SOL $\in]0;T_1[$
Undersaturated regime
$\rho_2 \leqslant \rho_1 \leqslant \bar{\rho} \leqslant \rho_3$
$T_2 \leqslant R\frac{\rho_c}{\rho_c - \rho_2}$
$\frac{R}{C} \leqslant \frac{(\rho_3 - \bar{\rho})(\rho_c - \rho_1)}{\rho_c(\rho_3 - \rho_1)}$
SOL $= 0$
Regime congested
$\rho_2 \leqslant \rho_1 \leqslant \bar{\rho} \leqslant \rho_3$
Undersaturated regime
$\rho_2 \leqslant \rho_1 \leqslant \bar{\rho} \leqslant \rho_3$
$T_2 \leqslant R\frac{\rho_c}{\rho_c - \rho_2}$
$\frac{R}{C} \geqslant \frac{(\rho_3 - \bar{\rho})(\rho_c - \rho_1)}{\rho_c(\rho_3 - \rho_1)}$
SOL $= T_1$
$\rho_2 \leqslant \bar{\rho} \leqslant \rho_1 \leqslant \rho_3$
$\rho_2 \leqslant \rho_1 \leqslant \bar{\rho} \leqslant \rho_3$
$T_2 \geqslant R\frac{\rho_c}{\rho_c - \rho_2}$

become important, a real-time control using sensors could work in such a case, because it would measure the output of an intersection and transmit to the upstream intersection what is its input. Given the input, the traffic light is able to apply the optimal control using the diagram on page 27. A limit for this is the time of computation, the sensors need the last car to leave the intersection before sending the data while the first car of the flow might have reached the next intersection before. The length of the link must not be too short then or we would have to figure out a way of measuring the densities with a small amount of data. This also assumes a huge investment at first because all the intersections should be equipped with sensors.

Chapter 4

Optimization taking into account the two ways of the traffic

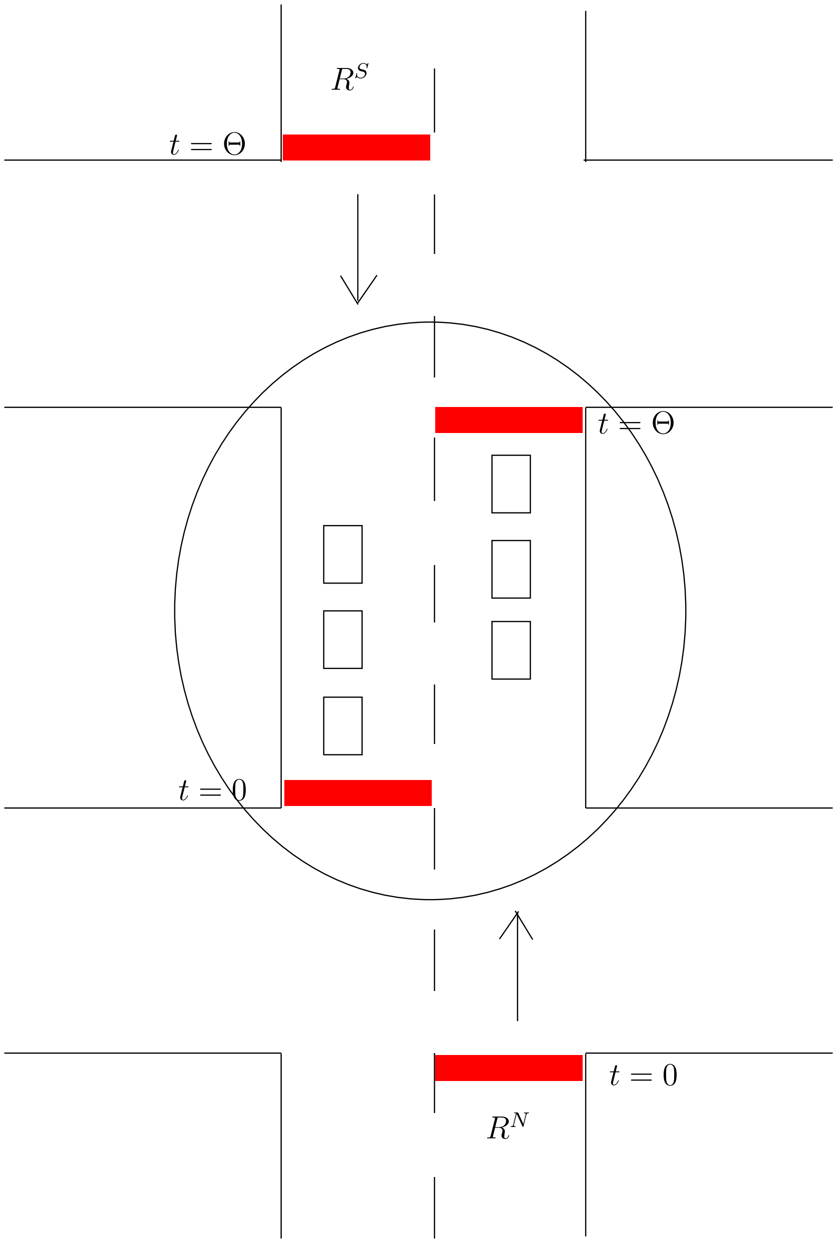
4.1 Context of the situation

Definition of the new cost function We saw earlier that optimizing one way of the traffic amounts to creating a green wave spreading along the traffic lights. This fact is due to the relative absence of constraints in our optimization problem. This absence of constraints is not realistic because the opposite direction of the traffic could be stuck without affecting the cost function from our previous model. A natural constraint to be added is a minimum flow on the opposite direction of the traffic, which can be translated by a maximum waiting time at the red lights. If we define $\Gamma_{Northway}$ and $\Gamma_{Southway}$ respectively as the total delay for the way heading north and the total delay for the way heading south, we write this constraint : $\Gamma_{Southway} < H$ where H is arbitrarily chosen depending on how much flow we chose to have on the other direction.

The problem is expressed in the same way than the previous one. We still have two red lights and the decision variable is the offset Θ, absolute this time. We cannot reduce the offset to a relative one as previously because the offset in the two directions are opposite. The figure 4.1 shows the described situation. The optimization program is thus the following:

$$\begin{aligned} &\min_{\Theta \in [0;C]} \Gamma_{Northway} \\ &\text{s.t. } \Gamma_{Southway} < H \end{aligned} \tag{4.1}$$

We use Lagrangian multipliers to get rid of the constraint, and it gives



the dual problem :

$$\max_{\lambda \geqslant 0} \min_{\Theta \in [0;C]} \Gamma_{Northway} + \lambda \Gamma_{Southway} \tag{4.2}$$

So, the main problem is to minimize a weighted sum of the waiting times on each direction. As this problem can be hard to solve, we decide to start with the special case where $\lambda = 1$ meaning we want to optimize the global system of the two ways without being to the advantage of one specific direction.

Simplifications In order to give the problem a particular structure making it easier to solve, we do not take into account in this part the incoming and outgoing flows from the side streets. Then, the North way platoons are $(R^N, 0), (\theta_c^N, \rho_c), (C - R^N - \theta_c^N, \rho_f^N)$ and the south way platoons are $(R^S, 0), (\theta_c^S, \rho_c), (C - R^S - \theta_c^S, \rho_f^S)$, where R is the duration of the red light, θ_c the queue dissipation duration, and ρ_f the density of the third platoon resulting from a previous merging between several platoons as explained in the chapter 2. The N and S mean the parameters belong to the North or the South way. A seventh parameter common to both sides is to be added to the model, the travel time tt from one traffic light to the other. Assuming all the vehicles move at the free flow speed v_f, $tt = \frac{L}{v_f}$, with L the length of the link between the two traffic lights.

As the previous problem, the set of solutions is discrete and the solution depends on several conditions over all the parameters. In order to reduce the degree of freedom of our problem and see some global cases where we can actually say some generality, we decide to focus during a first time on the case where the two ways are saturated. In this case, the two parameters ρ_f do not matter anymore because the third platoon is no longer existent, and we have the constraints $\theta_c = C - R$ on both sides meaning the queue takes the entire green time to discharge. Then, we have only three parameters left for this particular case.

4.2 Classification of the solutions

First observations After taking a look at some curves representing the total amount of delay in random situations, we can see an interesting fact. We get the minimum of the function on either tt or $C - tt$ meaning optimizing two ways is equivalent to optimize one way or another. The major difficulty is now to choose which way to optimize in order to actually get the same minimum as if we have two ways. Even though the minimum might be reached somewhere else, it seems more likely to have the same minimum as the one-way case.

Classification protocol To choose efficiently which way we have to optimize, we decide to classify the different cases. We generate random possible situations uniformly distributed and we plot them in a cut of the 3-D space of all the possible situations. The color we use to plot them depends on the

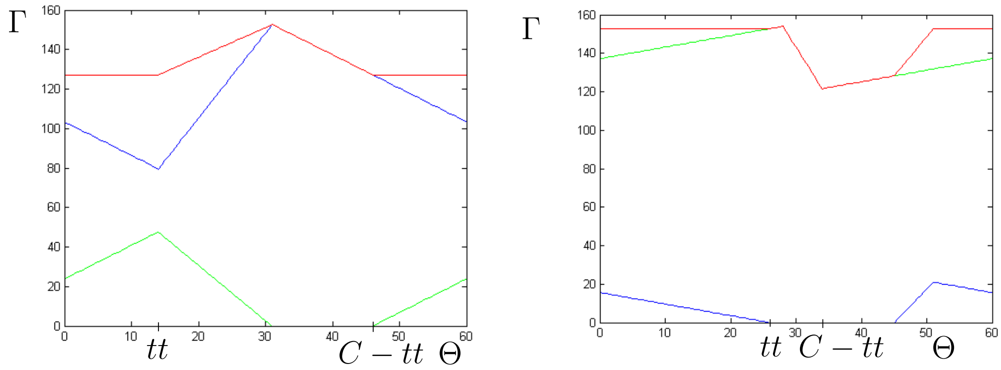


Figure 4.1: The blue curve is the delay of the north side, the green one is the delay of the south and the red one the total. On the left $R^N = 17, R^S = 32, tt = 14$ and on the right $R^N = 25, R^S = 6, tt = 26$

way we have to optimize. As we want to compare the parameters of the two ways, a natural cut to be used is the plane (R^N, R^S), tt being a parameter common to both ways. As a result, for a given value of tt, we obtain the following diagram in the figure 4.2.

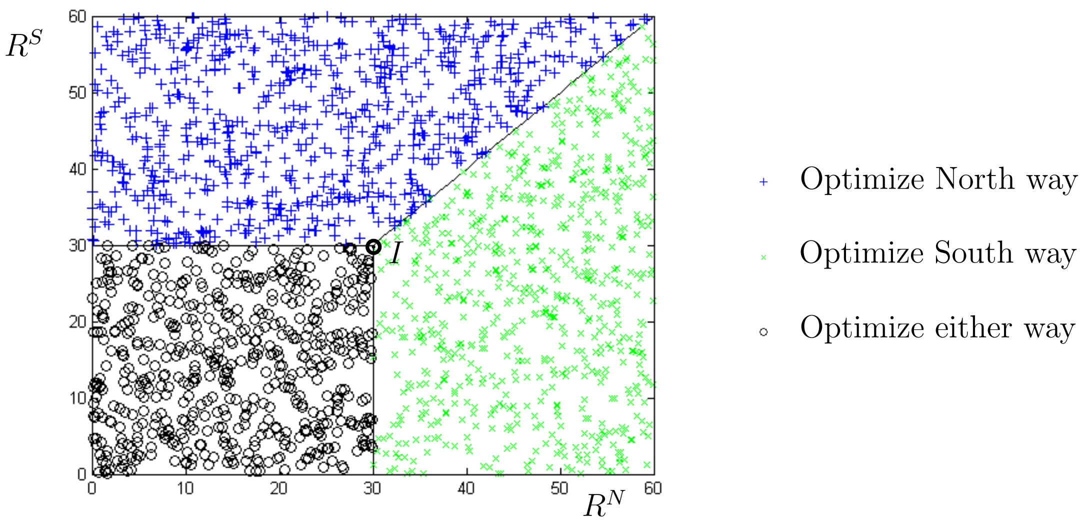


Figure 4.2: The blue + are the situations when we have to optimize the north way, the green x for the south way and the black o when it doesn't make any difference to choose between the two. Here, $tt = 15$.

We can see in the figure 4.2 three distinct areas separated by straight lines. We easily assume this bounds depend on tt, the last parameter, fixed in the previous diagram. Though, the diagram keeps its shape for different values of tt. The coordinates of the point I in the middle are the only thing changing with tt.

Equations of the frontiers We implemented a code using dichotomy to find the coordinates of I automatically for different values of tt. In fact, the code has just to seek the abscissa as I is always on the first bisectrix. Then, on the figure 4.3 is the curve of the variations of this abscissa x_I with tt.

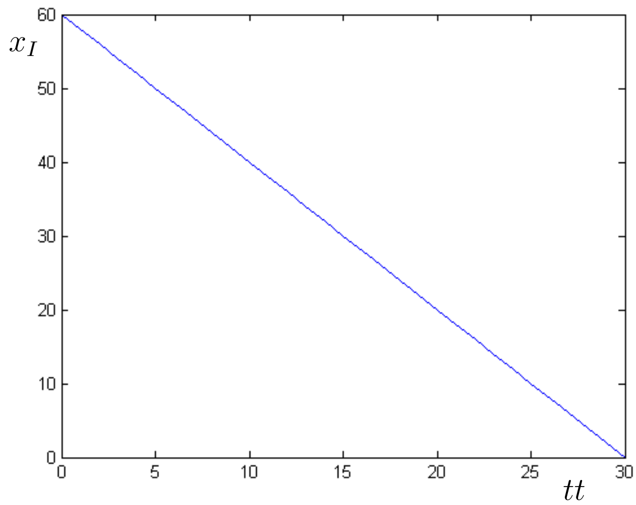


Figure 4.3: The abscissa of the point in function of tt

4.3 Analysis of the results

We derive from this graph that I has $(C - 2tt, C - 2tt)$ as coordinates. The three areas in this case are now well-defined.

We derive from this diagram that the best choice to do is to optimize the way with the longest red light duration. If we consider as an example $R^S \geqslant R^N$, it means the platoon at critical density on the North way has a duration $C - R^N$ longer than the green time $C - R^S$. Thus, some vehicles of this platoon must stop at the red light, whereas the situation is the opposite for the other direction. The platoon of critical density on the South way has a duration $C - R^S$ shorter than the green time $C - R^N$, this platoon can go through the intersection without any vehicle stopping. This platoon has more freedom in the choice of this offset and the gap between the optimal offset and another offset has less influence than on the other direction in which some vehicles must stop anyway. So, the optimal choice is to set the offset with respect to the way equipped with the longest red time.

This case is the easiest one we can study for this general situation. A lot of work is possible to improve the results. To add complexity to this problem, we planned to study the effect of making one way out of the two undersaturated, but two new parameters are to study then, the queue discharge duration and the density of the third platoon for the undersaturated way. We could not study it because we lack of time, but the primary results show us the areas are worse-defined - the frontiers are not straight lines anymore - and the optimization of the two ways is not equivalent to optimize only one way, but sometimes both are to be taken into account. Once this work is done, several intersections should be studied but, thanks to the similar structure of the input and the output, the results should be derived quite easily.

Chapter 5

Conclusion

This work is mainly an introduction to the model of the intersection which can be considered as a system or a projection on a subset. This model allows the traffic flow to be characterized by a small and finite amount of parameters. Moreover, the study of several intersections one after another is made easier by the similar structure of the input and the output of the intersection.

This model, allows a analytical solution to the classic problem of the traffic lights coordination. The limited number of parameters and the structure of the traffic flow make the function representing the total waiting time of the vehicles during a cycle quasi-convex and give the exact position of the minimum of this function. As the model was built to fit in a study of several consecutive intersections, the optimization of a single intersection was just a step and gives very naturally a behaviour and jumping conditions from one intersection to another.

This behaviour shows that the optimal offset to be implemented remains the same after a few intersections and gives birth to a green wave. The green wave being an intuitive way to optimize the offset with a several intersections, it means the model is viable and gives results which make sense even though the theory seems quite heavy compared to the results provided. Nevertheless, the results are much more complete than just giving a green wave and could be implemented in a real-time control system. Some sensors would measure the outgoing flow of an intersection and would send these informations to the downstream traffic light, giving him all the parameters to compute the optimal offset. Such a system should be analysed more precisely before implementation because it presents several issues. The investment for sensors at each intersection could be huge as the computation time could be longer than the flow travel time from one intersection to another.

This model is not limited to the one-way synchronization problem and could be applied to model the flow in a lot of arterial traffic situations. We intend to study the two-ways case even though we finally got partial results due to a lack of time. With these partial results, we manage to find that, in a congested regime, optimizing one way is equivalent to optimize the way getting the longer red light duration. This result does not hold when the regime become undersaturated but gives some behaviour of the system which can be adapted to further studies of the two-ways problem.

Apart from traffic lights synchronization, this model could be used in bigger projects such as travel time estimation along a urban road going through several intersections. This problem is studied nowadays by transportation laboratories in order to improve traffic management inside the cities and not only on the highways.

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Bibliography

[1] A. Bayen A. Hofleitner, R. Herring. A hydrodynamic theory based statistical model of arterial traffic. *California Center for Innovative Transportation Research Report*, January 2001.

[2] Carlos F. Daganzo. The cell transmission model: A dynamic representation of highway traffic consistent with the hydrodynamic theory. *Transportation Research Part B: Methodological*, 28(4):269 – 287, 1994.

[3] L. C. Evans. Graduate studies in mathematics, vol. 19. *Partial Differential Equations*, 1998.

[4] N. H. Gartner and C. Stamatiadis. Arterial-based control of traffic flow in urban grid networks. *Mathematical and Computer Modelling*, 35(5-6):657 – 671, 2002.

[5] N. Geroliminis and A. Skabardonis. Prediction of arrival profiles and queue lengths along signalized arterials by using a markov decision process. *Transportation Research Record : Journal of the Transportation research Board*, 2005.

[6] S. Lämmer, R. Donner, and D. Helbing. Anticipative control of switched queueing systems. *The European Physical Journal B - Condensed Matter and Complex Systems*, 63:341–347, 2008. 10.1140/epjb/e2007-00346-5.

[7] M. Lighthill and G. Whitham. On kinematic waves. ii. a theory of traffic flow on long crwaded roads. *Proceedings of the Royal Society of London*, Series A, Mathematical and Physical Sciences:229(1178):317–345, May 1955.

[8] T. A. Litman. Transportation cost and benefit analysis ii - congestion cost. *Victoria Transport Policy Institute*, January 2009.

[9] D. Bullock M. Abbas and L. Head. Real-time offset transitioning algorithm for coordinating traffic signals. *Transportation Research Record*

: Journal of the Transportation Research Board, Volume 1748:26–39, January 2001.

[10] P. Richards. Shock waves on the highway. *Operations Research*, page 14(4), February 1956.

[11] D. I. Robertson. Transyt - a traffic network study tool. *Report No TRRL-LR-253 (Transport and Road Research Laboratory, Crowthorne)*, 1969.

[12] D.I. Robertson and R.D. Bretherton. Optimizing networks of traffic signals in real time-the scoot method. *Vehicular Technology, IEEE Transactions on*, 40(1):11 –15, feb 1991.

[13] D. Schrank and T. Lomax. Urban mobility study. *Texas Transportation Institute*, 2007.