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# Reconstruction of boundary conditions from internal conditions using viability theory

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*Abstract*— **This article presents a method for reconstructing downstream boundary conditions to a HamiltonJacobi partial differential equation for which initial and upstream boundary conditions are prescribed as piecewise affine functions and an internal condition is prescribed as an affine function. Based on viability theory, we reconstruct the downstream boundary condition such that the solution of the Hamilton-Jacobi equation with the prescribed initial and upstream conditions and reconstructed downstream boundary condition satisfies the internal value condition.**

**This work has important applications for estimation in flow networks with unknown capacity reductions. It is applied to urban traffic, to reconstruct signal timings and temporary capacity reductions at intersections, using Lagrangian sensing such as GPS devices onboard vehicles.**

## I. Introduction

The computation of numerical solutions to the *Hamilton-Jacobi* (HJ) *partial differential equation* (PDE) subject to boundary conditions, initial conditions or sometimes terminal conditions is a topic which has generated significant interest in the control and numerical analysis community [18], [16], [14]. However, the integration of initial or boundary conditions alone may not be sufficient to solve new data reconstruction problems arising in the context of Lagrangian sensing [11]. We consider the specific problem of reconstructing boundary conditions from internal value conditions provided by Lagrangian sensing. This problem has important practical applications, in particular in the context of flow networks for which it provides information on decreases in the capacity, which is important to detect and control saturation and bottlenecks before they propagate throughout the network.

The fundamental challenge of integrating these different types of sensing data is the proper use of a constitutive model of the system. A model capable of mathematically handling initial, boundary and internal conditions for the HJ–PDE is presented in [5], [6].

An application of interest is the design of accurate real time traffic monitoring systems on arterial networks [3], [19], [10]. The physics of traffic flow is governed by the presence of signals, with, in general, unknown parameters, which lead to periodic drops of the capacity at intersections and to the formation of queues. Today, the GPS technology provides Lagrangian measurements (happening onboard the vehicle) of traffic conditions which can be used to reconstruct downstream boundary conditions, *i.e.* to estimate capacity drops. The state of the road network (density, velocity and flow) can then be estimated at any location $x$ and time $t$.

The article is organized as follows. In Section II, we introduce the mathematical background and state the reconstruction problem of the downstream boundary condition. In Section III, we prove the existence of a solution to the reconstruction problem under some compatibility conditions between the given initial, upstream and internal value conditions. We detail an algorithm to solve the reconstruction problem in Section IV and illustrate it numerically in Section V.

## II. Problem statement

### A. Mathematical background

We investigate the solution to the following HJ–PDE on the domain $(t,x) \in [0, t_{\max}] \times [\xi, \chi]$, sometimes known as the *Moskowitz HJ-PDE* [15], [8].

$$\frac{\partial \mathbf{M}(t,x)}{\partial t} - \psi\left(-\frac{\partial \mathbf{M}(t,x)}{\partial x}\right) = 0 \tag{1}$$

The function $\psi$, called *Hamiltonian*, is assumed to be concave on its domain of definition $D_\psi = [0, \rho_{\max}]$ and to satisfy $\psi(0) = \psi(\rho_{\max}) = 0$. We call $q_{\max}$ the maximum value of $\psi$ on $D_\psi$ and define $\nu^\flat = \psi'(0)$ and $\nu^\sharp = -\psi'(\rho_{\max})$. The concavity and the condition that $\psi(0) = \psi(\rho_{\max}) = 0$ impose that $\nu^\flat > 0$ and $\nu^\sharp > 0$.

The proper notion of solution to (1) with initial and boundary condition is well studied in the literature [7]. However, the mathematical properties of the solution of (1) require specific treatments when we introduce internal boundary conditions. We use a specific control framework based on the use of Lax-Hopf's formula and viability theory [2] to add this type of conditions. We first define the convex transform $\varphi^*$ of the Hamiltonian as follows.

*Definition 1 (Convex transform):* Let $\psi$ be a concave function defined on $D_\psi$, its convex transform $\varphi^*$ takes finite values on $D_{\varphi^*} = [-\nu^\flat, \nu^\sharp]$:

$$\varphi^*(u) = \begin{cases} \sup\limits_{p \in D_\psi} [pu + \psi(p)] & \text{if } u \in [-\nu^\flat, \nu^\sharp] \\ +\infty & \text{otherwise} \end{cases} \tag{2}$$

Let $\mathbf{c}$ be a lower semi-continuous function defined on a subset of $[0, t_{\max}] \times [\xi, \chi]$. It represents a *value condition*, *i.e.* a value that we want to impose on the solution of (1).

*Proposition 1 (Lax-Hopf formula):* The viability epi-solution [1], [5] $\mathbf{M_c}$ associated with $\mathbf{c}$ is given by

$$\mathbf{M_c}(t,x) = \inf_{(u,T) \in D_{\varphi^*} \times \mathbb{R}^+} [\mathbf{c}(t-T, x+Tu) + T\varphi^*(u)] \tag{3}$$

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It is the unique generalized solution of (1) in the *Barron-Jensen/Frankowska* (B-J/F) sense [1] associated with $\mathbf{c}$. Equation (3) also implies an inf-morphism property [1], [5], [6], which is a key property used to develop the algorithms of this article.

*Proposition 2 (Inf-morphism):* Let $\mathbf{c}$ be the minimum of a finite number of functions $\mathbf{c}_i$, $i \in I$. The viability episolution $\mathbf{M}_{\mathbf{c}}$ defined by (3) can be written as:

$$\forall\,(t,x) \in [0,t_{\max}] \times [\xi,\chi]\ \mathbf{M}_{\mathbf{c}}(t,x) = \inf_{i\in I} \mathbf{M}_{\mathbf{c}_i}(t,x)$$

The inf-morphism property is a practical tool to integrate new value conditions and separate a complex problem involving multiple value conditions into a set of more tractable sub-problems [5], [6].

### B. State estimation with affine initial and upstream boundary conditions

In this article, we assume that we are given continuous piecewise affine initial and upstream boundary conditions, denoted $\mathcal{M}_0$ and $\gamma$ respectively. We define affine functions $\mathcal{M}_{0_i}$, $i \in \{1,\dots,I_0\}$ and $\gamma_j$, $j \in \{1,\dots,I_\gamma\}$ such that $\forall (t,x) \in [0,t_{\max}] \times [\xi,\chi]$, $\mathcal{M}_0(t,x) = \min_{i=1}^{I_0} \mathcal{M}_{0_i}(t,x)$ and $\gamma(t,x) = \min_{j=1}^{I_\gamma} \gamma_j(t,x)$.

The Lax-Hopf formula (3) leads to an analytical expression of the solution associated with an affine value condition [6], omitted in this article for brevity. We introduce the following notation and definitions, referring to [6] for the proof of their existence.

*Definition 2 (Upper critical density $\overline{\rho}_c$):* For $\rho \in [0,\rho_{\max}]$, we define $\overline{\rho}_c$ as the largest solution of $\psi(\rho) = q_{\max}$.

*Definition 3 (Congested density associated with q [9]):* For $q \in [0,q_{\max}]$ we define $\overline{\rho}(q)$ as the unique solution of $\psi(\rho) = q$ for $\rho \in [\overline{\rho}_c, \rho_{\max}]$.

Following [4], we define the subderivative $\partial_-$ and the superderivative $\partial_+$ as follows:

$$v \in \partial_- f(x_0) \Leftrightarrow \forall x \in D_f,\ f(x) \geq f(x_0) + v(x-x_0)$$
$$v \in \partial_+ f(x_0) \Leftrightarrow \forall x \in D_f,\ f(x) \leq f(x_0) + v(x-x_0)$$

*Definition 4:* For $\rho \in [0,\rho_{\max}]$, we define $u_0^+(\rho)$ as an element of $-\partial_+\psi(\rho) \cap \mathbb{R}^+$. Note that $u_0^+(\rho)$ is not uniquely defined if $\psi$ is not differentiable in $\rho$. However, the specific choice of $u_0^+(\rho) \in -\partial_+\psi(\rho)$ does not influence the results derived later in this article.

*Definition 5 (Capture time $\overline{T}_0$):* The function $\overline{T}_0$ is defined as follows:

$$\overline{T}_0(\rho,x) = \begin{cases} \frac{\chi - x}{u_0^+(\rho)} & \text{if } u_0^+(\rho) \neq 0 \\ +\infty & \text{otherwise} \end{cases} \quad \forall (\rho,x) \in [\overline{\rho}_c, \rho_{\max}] \times [\xi,\chi],$$

### C. Problem statement

We assume that, besides the piecewise affine initial and upstream boundary conditions, we are given an affine internal value condition $\mu$, defined as follows:

*Definition 6 (Affine internal value condition $\mu$):* For $t \in [t_1,t_2]$, we define $\zeta(t) = x_1 + v(t-t_1)$. The function $\mu$ reads

$$\mu(t,x) = \begin{cases} g(t-t_1)+h & \text{if } t \in [t_1,t_2] \text{ and } x = \zeta(t) \\ +\infty & \text{otherwise} \end{cases} \tag{4}$$

We assume that the constants $g$ and $v$ satisfy $0 \leq g \leq \psi(\overline{\rho}_c) - \overline{\rho}_c v$ and $0 \leq v \leq \nu^\flat$.

We call downstream boundary condition $\beta$, a value condition that takes finite values on a subset of $[0,t_{\max}] \times \{\chi\}$. At time $t$, the downstream boundary condition $\beta(t,\chi)$ provides information on decreases in the capacity at $x = \chi$, which is important to detect and control saturation and bottlenecks before they propagate throughout the network. This motivates the following reconstruction problem:

*Problem 1 (Initial Boundary Value Problem):* We are given an affine internal value condition $\mu$, piecewise affine upstream boundary condition $\gamma$ and initial condition $\mathcal{M}_0$. We want to *reconstruct* the downstream boundary condition $\hat{\beta}$ such that the B-J/F solution of the *Initial Boundary Value Problem* of the HJ-PDE (1) with the prescribed initial and boundary conditions $\mathcal{M}_0$, $\gamma$ and $\hat{\beta}$ satisfies the internal condition:

$$\forall t \in [t_1,t_2],\ \forall x = \zeta(t),$$
$$\min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma, \mathbf{M}_{\hat{\beta}})(t,x) = \mu(t,x). \tag{5}$$

We define an affine downstream boundary solution $\beta_k$ as follows:

$$\beta_k(t,x) = \begin{cases} f_k + e_k(t-\overline{\beta}_k) & \text{if } t \in [\overline{\beta}_k, \overline{\beta}_{k+1}] \text{ and } x = \chi \\ +\infty & \text{otherwise} \end{cases} \tag{6}$$

We define $\rho_k = \overline{\rho}(e_k)$ (Definition 3). The expression of the solution of the HJ-PDE subject to the downstream boundary condition $\beta_k$ is denoted $\mathbf{M}_{\beta_k}$ and can be computed explicitly [6]. We define the domains $(i)$, $(ii)$ and $(iii)$ in which the solution has a specific analytical expression:

$$\mathbf{M}_{\beta_k}(t,x) = \begin{cases} f_k + (t-\overline{\beta}_k)\varphi^*\left(\frac{\chi - x}{t-\overline{\beta}_k}\right) \text{ if } \overline{T}_0(\rho_k,x) \geq t - \overline{\beta}_k & (i) \\ (t-\overline{\beta}_k)e_k + (\chi - x)\rho_k + f_k & (ii) \\ \qquad \text{if } \overline{T}_0(\rho_k,x) \in [t-\overline{\beta}_{k+1}, t-\overline{\beta}_k] & \\ f_k + (\overline{\beta}_{k+1} - \overline{\beta}_k)e_k + (t-\overline{\beta}_{k+1})\varphi^*\left(\frac{\chi - x}{t-\overline{\beta}_{k+1}}\right) & \\ \qquad \text{if } \overline{T}_0(\rho_k,x) \leq t - \overline{\beta}_{k+1} & (iii) \end{cases} \tag{7}$$

## III. Existence of a downstream boundary condition

In this section, we derive conditions on $\mathcal{M}_0$, $\gamma$ and $\mu$ for the existence of a downstream boundary condition $\hat{\beta}$ which solves Problem 1. We study uniqueness properties among piecewise affine solutions and exhibit a solution that corresponds to a constant limitation of the maximum flow in an interval $[\tau_1,\tau_2]$.

### A. Interval in which the downstream boundary condition is affine

Given that $\mu$ is affine, $\hat{\beta}$ is necessarily such that $\mathbf{M}_{\hat{\beta}}$ is affine on the trajectory $\zeta$ (since $\mathbf{M}_{\hat{\beta}}$ and $\mu$ coincide on the domain of $\mu$). The derivative of $\mathbf{M}_{\hat{\beta}}$ in the direction $(1,v)$ should thus exist in the domain $\{(t,x) \text{ s.t. } t \in [t_1,t_2],\ x = \zeta(t)\}$ and should be equal to $g$. We first introduce the following lemma:

*Lemma 1 (Intervals in which $\varphi^*$ is affine):* The function $\varphi^*$ is affine in $[u_1,u_2]$ if and only if there exists $\rho \in D_\psi$ such that $(u_1,u_2) \subset -\partial^+\psi(\rho)$.

*Proof:* The function $\varphi^*$ is affine in the interval $[u_1, u_2]$ if and only if its subgradient is reduced to a given $\rho^*$ in $(u_1, u_2)$. The subderivative of $\varphi^*$ satisfies the Legendre-Fenchel inversion formula [1]:

$$u \in -\partial_+\psi(\rho) \Leftrightarrow \rho \in \partial_-\varphi^*(u).$$

Since $\partial_-\varphi^*(u) = \{\rho^*\}$ for $u \in (u_1, u_2)$, we have $(u_1, u_2) \subset -\partial^+\psi(\rho^*)$. ■

*Definition 7 (Density associated with $v$ and $g$):* Let $f_v$ be defined for $\rho \in [0, \rho_{\max}]$ by $f_v(\rho) = \psi(\rho) - v\rho$. The function is concave as the sum of concave functions, and attains its maximum value $\varphi^*(-v)$ in a closed interval (Definition 1). Let $\rho^*$ be the upper bound of this interval. We assumed in Definition 6 that $0 \le g \le \psi(\overline{\rho}_c) - \overline{\rho}_c v = f_v(\overline{\rho}_c)$, and thus $0 \le g \le \varphi^*(-v)$. Since $f_v$ is continuous and $f_v(\rho_{\max}) \le 0$, the intermediate value theorem states that there exists a solution $\tilde{\rho}(v, g) \in [\rho^*, \rho_{\max}]$ such that $f_v(\tilde{\rho}(v, g)) = g$. Given that $f_v$ is concave and given the definition of $\rho^*$, $f_v$ is strictly decreasing on $[\rho^*, \rho_{\max}]$ which proves that $\tilde{\rho}(v, g)$ is unique. Given that $g \le f_v(\overline{\rho}_c)$, we have $\tilde{\rho}(v, g) \ge \overline{\rho}_c$.

*Definition 8 (Compatibility conditions):* A necessary condition for Problem (1) to be well posed is to have *compatible* initial, upstream and internal conditions [5], [6]. This means that all these conditions can be imposed simultaneously and is written as

$$\begin{array}{l} \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma)(t,x) \ge \mu(t,x) \ \forall t \in [t_1, t_2],\ x = \zeta(t) \\ \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\mu)(t,x) \ge \gamma(t,x) \ \forall (t,x) \in [0, t_{\max}] \times \{\xi\} \\ \min(\mathbf{M}_\gamma, \mathbf{M}_\mu)(t,x) \ge \mathcal{M}_0(t,x) \ \forall (t,x) \{0\} \times [\xi, \chi] \end{array} \tag{8}$$

In the remainder of this article, we call $\rho_{\text{out}} = \tilde{\rho}(v, g)$ and $q_{\text{out}} = \psi(\rho_{\text{out}})$ and assume that the compatibility conditions between $\mathcal{M}_0$, $\gamma$ and $\mu$ are satisfied.

*Proposition 3 (Affine boundary condition):* If the internal condition $\mu$ is such that $\psi$ is differentiable at $\rho_{\text{out}}$, there exists an interval $[\tilde{\tau}_1, \tilde{\tau}_2]$ in which any piecewise affine solution of Problem 1 is necessarily affine, with temporal derivative equal to $q_{\text{out}}$.

*Proof:* We consider a potential piecewise affine solution $\hat{\beta}$. If such a solution exists, there exists a set of functions $(\beta_k)_{k \in K}$, defined by (6) such that $\forall (t, x) \in [0, t_{\max}] \times \{\chi\}$, $\hat{\beta}(t, x) = \min_{k \in K} \beta_k(t, x)$. For each $k \in K$, we consider the domain defined by $t \in [\underline{t}_k, \overline{t}_k]$ and $x = \zeta(t)$ in which $\mathbf{M}_{\hat{\beta}}(t, x) = \mathbf{M}_{\beta_k}(t, x)$. We show that the points $(\underline{t}_k, \zeta(\underline{t}_k))$ and $(\overline{t}_k, \zeta(\overline{t}_k))$ necessarily belong to the domain $(ii)$ of the downstream boundary condition $\beta_k$. We then show that the temporal derivative of $\beta_k$ is necessarily equal to $q_{\text{out}}$ and conclude the proof.

Since $\mathbf{M}_{\hat{\beta}}$ is a solution of Problem 1, it takes finite values at $(\overline{t}_k, \zeta(\overline{t}_k))$ and $(\underline{t}_k, \zeta(\underline{t}_k))$. These points necessarily belong to one of the domains $(i)$, $(ii)$ or $(iii)$ of $\mathbf{M}_{\beta_k}$.

• If $(\overline{t}_k, \zeta(\overline{t}_k))$ belongs to domain $(iii)$, we define $\delta_k \ge \underline{t}_k$, the first time such that $(\delta_k, \zeta(\delta_k))$ is in domain $(iii)$. The function $\mathbf{M}_{\beta_k}$ is necessarily affine along the trajectory $\zeta$ with derivative equal to $g$. For any $t \in [\delta_k, \overline{t}_k]$ such that $\mathbf{M}_{\beta_k}$ is differentiable in $(t, \zeta(t))$, its total derivative along the trajectory $\zeta$ is given by

$$\frac{d\mathbf{M}_{\beta_k}}{dt}(t, \zeta(t)) = \varphi^*(u(t)) - (v + u(t))\,(\varphi^*)'(u(t)), \tag{9}$$

with $u(t) = \frac{\chi - \zeta(t)}{t - \overline{\beta}_{k+1}}$. We then have

$$\frac{d^2\mathbf{M}_{\beta_k}}{dt^2}(t, \zeta(t)) = 0 \Leftrightarrow (\varphi^*)''(u(t)) = 0,\ \forall t \in [\delta_k, \overline{t}_k].$$

Necessarily, $\varphi^*$ is affine on $[u(\delta_k), u(\overline{t}_k)]$ and Lemma 1 proves that there exists $\rho^* \in D_\psi$ such that $[u(\delta_k), u(\overline{t}_k)] \subset -\partial^+\psi(\rho^*)$. It implies that, on the trajectory $\zeta$, $\varphi^*(u(t)) = \psi(\rho^*) + u(t)\rho^*$ and $(\varphi^*)'(u(t)) = \rho^*$. The total derivative of $\mathbf{M}_{\beta_k}$ along the trajectory is thus given by

$$\frac{d\mathbf{M}_{\beta_k}}{dt}(t, \zeta(t)) = \psi(\rho^*) - v\rho^*.$$

Since $\frac{d\mathbf{M}_{\beta_k}}{dt}(t, \zeta(t)) = g$, $\rho^* = \tilde{\rho}(v, g)$; since $\psi$ is differentiable at $\rho_{\text{out}} = \tilde{\rho}(v, g)$, $-\partial^+\psi(\rho^*)$ is reduced to a singleton. This implies that $u(\delta_k) = u(\overline{t}_k)$ and thus $\delta_k = \overline{t}_k$. The point $(\overline{t}_k, \zeta(\overline{t}_k))$ is at the boundary of the domains $(ii)$ and $(iii)$. Similarly, if $(\underline{t}_k, \zeta(\underline{t}_k))$ is in the domain $(i)$, it is also in the domain $(ii)$ and thus at the intersection of the two domains.

• In the domain $(ii)$, $\mathbf{M}_{\beta_k}$ is affine and its total derivative along the trajectory $\zeta$ is given by

$$\frac{d\mathbf{M}_{\beta_k}}{dt}(t, \zeta(t)) = \psi(\rho_k) - v\rho_k.$$

Necessarily, $\rho_k = \rho_{\text{out}}$ and $f_k = \psi(\rho_k) = q_{\text{out}}$. For the points $(\underline{t}_k, \zeta(\underline{t}_k))$ and $(\overline{t}_k, \zeta(\overline{t}_k))$ to be included in the domain $(ii)$, we have

$$\overline{\beta}_k \le \underline{t}_k - \frac{\chi - \zeta(\underline{t}_k)}{u_0^+(\rho_{\text{out}}, \zeta(\underline{t}_k))} \text{ and } \overline{\beta}_{k+1} \ge \overline{t}_k - \frac{\chi - \zeta(\overline{t}_k)}{u_0^+(\rho_{\text{out}}, \zeta(\overline{t}_k))}$$

For all $k$ such that $\mathbf{M}_{\hat{\beta}}(t, \zeta(t)) = \mathbf{M}_{\beta_k}(t, \zeta(t))$ for $t \in [\underline{t}_k, \overline{t}_k]$, $\beta_k$ has a temporal derivative equal to $q_{\text{out}}$. The continuity of $\mathbf{M}_{\hat{\beta}}$ imposes that there exists a unique $k = k^*$ such that $\mathbf{M}_{\hat{\beta}}(t, x) = \mathbf{M}_{\beta_{k^*}}(t, x)$ on the trajectory $\zeta$. We define $\tilde{\tau}_1$ and $\tilde{\tau}_2$ as follows:

$$\tilde{\tau}_1 = t_1 - \frac{\chi - x_1}{u_0^+(\rho_{\text{out}}, x_1)} \text{ and } \tilde{\tau}_2 = t_2 - \frac{\chi - x_2}{u_0^+(\rho_{\text{out}}, x_2)}. \tag{10}$$

The boundary condition $\beta_{k^*}$ takes finite values in a domain including $[\tilde{\tau}_1, \tilde{\tau}_2] \times \{\chi\}$ in which its temporal derivative is equal to $q_{\text{out}}$.

If $\psi$ is not differentiable in $\rho_{\text{out}}$, the choice of $\beta_{k^*}$ also leads to the equality of the derivatives of $\mu$ and $\mathbf{M}_{\beta_{k^*}}$ on the trajectory $\zeta$, even though this choice may no longer be unique. ■

### B. Existence under compatibility conditions

*Proposition 4 (Existence):* If the internal value condition $\mu$ is affine, if the initial and upstream boundary conditions are piecewise affine and if the feasibility conditions of Definition 8 are satisfied, there exists a downstream boundary condition $\hat{\beta}$ solution of Problem 1. We exhibit a solution which is affine on the smallest interval $[\tau_1, \tau_2] \supset [\tilde{\tau}_1, \tilde{\tau}_2]$, representing a constant limitation of the maximum flow.

*Proof:* We search for a potential solution $\hat{\beta}$ of Problem 1 which represents a constant limitation of the maximum flow during a time interval $[\tau_1, \tau_2]$. To achieve this goal, we search for $\tau_1 \le \tilde{\tau}_1$, $\tau_2 \ge \tilde{\tau}_2$, $m$ and $\rho$ such that $\hat{\beta}(t, \chi) = m + (t - \tau_1)\psi(\rho)$, $\forall t \in [\tau_1, \tau_2]$. We call $\hat{\beta}^*$ the restriction of $\hat{\beta}$ in $[\tau_1, \tau_2] \times \{\chi\}$ and $\mathbf{M}_{\hat{\beta}^*}$ the associated viability episolution. For $t \le \tau_1$, there is no downstream constraint in the flow and we choose $\hat{\beta}(t, \chi) = \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma)(t, \chi)$. For $t \ge \tau_2$, there is no limitation of the maximum flow at $x = \chi$. The

flow at $x = \chi$ is given by $\min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma, \mathbf{M}_{\hat{\beta}^*})(t, \chi)$, it depends on the upstream, initial and upstream boundary condition $\hat{\beta}^*$.

From the results of Proposition 3, we choose $\rho = \rho_{\text{out}}$. With this choice, the trajectory $\zeta$ is included in the domain $(ii)$ of $\mathbf{M}_{\hat{\beta}^*}$. The function $\mathbf{M}_{\hat{\beta}^*}$ is affine in domain $(ii)$ and its derivative along the trajectory $\zeta$ is equal to $g$.

• *Equation satisfied by $\tau_1$ and $\tau_2$*: In the domain $(ii)$, we have $\mathbf{M}_{\hat{\beta}^*}(t, x) = (t - \tau_1)q_{\text{out}} + (\chi - x)\rho_{\text{out}} + m$ and we want $\mathbf{M}_{\hat{\beta}^*}(t_1, x_1) = \mu(t_1, x_1) = h$. This condition imposes a relation between $\tau_1$, $m$ and $h$:

$$(t_1 - \tau_1)q_{\text{out}} + (\chi - x_1)\rho_{\text{out}} + m = h. \tag{11}$$

We define the function $\tilde{h}$ for $t \in [0, t_{\max}]$ by $\tilde{h}(t) = h - \rho_{\text{out}}(\chi - x_1) + (t - t_1)q_{\text{out}}$. With this notation, (11) is written $m = \tilde{h}(\tau_1)$. The continuity of $\hat{\beta}$ at $(\tau_1, \chi)$ imposes that $m = \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma)(\tau_1, \chi)$ which leads to the following equation for $\tau_1 \in [0, \tilde{\tau}_1]$

$$\tilde{h}(\tau_1) - \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma)(\tau_1, \chi) = 0. \tag{12}$$

We choose $\tau_2 = \tilde{\tau}_2$. Note that larger values of $\tau_2$ are possible, leading to a longer limitation of the maximum flow at $x = \chi$ but we choose the smallest solution. The observation of $\mu$ only provides a lower bound for the value of $\tau_2$. If Equation (12) has a solution in the interval $[0, \tilde{\tau}_1]$, we call $\tau_1$ the largest such solution and present an algorithm to compute this solution in Section IV. Otherwise, we set $\tau_1 = 0$ and we introduce specific feasibility conditions in Definition 8. ■

*Proposition 5 (Feasibility conditions):* The search for a piecewise affine limitation of the maximum flow implies that $\beta^*(t, \chi) \leq \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma)(t, \chi), \forall t \in [\tilde{\tau}_1, \tilde{\tau}_2]$ *i.e.*

$$\forall t \in [\tilde{\tau}_1, \tilde{\tau}_2],\ \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma)(t, \chi) \geq \tilde{h}(t). \tag{13}$$

If there is no solution to (12) in $[0, \tilde{\tau}_1]$, the feasibility conditions require the existence of $\hat{x} \in [\xi, \chi]$ such that the spatial derivative of $\mathcal{M}_0$ is $-\rho_{\text{out}}$ for $(t, x) \in \{0\} \times [\hat{x}, \chi]$ and such that $\mathcal{M}_0(0, \hat{x}) = h - (\hat{x} - x_1)\rho_{\text{out}} - t_1 q_{\text{out}}$.

If these conditions are satisfied, the construction of $\rho_{\text{out}}$, $\tau_2$, $\tau_1$ and $m$ leads to a solution of Problem 1.

*Proof:* This is true by construction. We define $\hat{\beta}^*(t, \chi) = m + (t - \tau_1)\psi(\rho_{\text{out}})$ for $t \in [\tau_1, \tau_2]$, the solution $\hat{\beta}$ takes finite values in $[0, t_{\max}] \times \{\chi\}$:

$$\hat{\beta}(t, x) = \begin{cases} \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma)(t, \chi) \text{ if } t \leq \tau_1 \\ \hat{\beta}^*(t, \chi) \text{ if } t \in [\tau_1, \tau_2] \\ \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma, \mathbf{M}_{\hat{\beta}^*})(t, \chi) \text{ if } t \geq \tau_2 \end{cases}.$$

The function defined as the minimum of $\mathbf{M}_{\mathcal{M}_0}$, $\mathbf{M}_\gamma$ and $\mathbf{M}_{\hat{\beta}}$ in the domain $[0, t_{\max}] \times [\xi, \chi]$ is a solution of the HJ-PDE (1). The compatibility conditions ensure that the boundary conditions are satisfied and the construction of $\hat{\beta}$ ensures that the function takes the same values as the internal condition $\mu$ for all $(t, x)$ on the trajectory defined by $\zeta$. ■

## IV. Reconstruction of downstream boundary conditions using one affine internal value condition

In this section, we present an algorithm which computes the largest solution of (12) in the interval $[0, \tilde{\tau}_1]$ or proves that there is no solution on this interval. The algorithm leverages the inf-morphism property (Proposition 2) and the convexity of $\mathbf{M}_{\mathbf{c}_i}$ for any convex target function $\mathbf{c}_i$ [6].

*Proposition 6 (Algorithm to compute $\tau_1$):* If (12) has a solution in $[0, \tilde{\tau}_1]$, its largest solution can be computed by solving a finite number of scalar convex optimization programs and scalar linear equations (Algorithm 1). If there is no solution in $[0, \tilde{\tau}_1]$, the same algorithm provides a proof that no solution exists.

**Algorithm 1** Algorithm for computing $\tau_1$

Define $\underline{\kappa}_i$, $\kappa_i^1$ and $\kappa_i^2$ for $i \in \{1, \ldots, I_0 + I_\gamma\}$,
$\mathcal{K} = \cup_i \{\underline{\kappa}_i, \kappa_i^1, \kappa_i^2\}$, $\tau_{\max} = t_1 - \frac{\chi - x_1}{u_0(\rho_{\text{out}})}$,
$\tau_{\min} = \max\{[0, \tau_{\max}) \cap \mathcal{K}\}$, $T = \emptyset$.
**while** $T == \emptyset$ **do**
  $I = \{i \in \{1, \ldots, I_0 + I_\gamma\} : \underline{\kappa}_i \leq \tau_{\max}\}$
  **for** $i \in I$ **do**
    $\underline{n}_i = \mathbf{M}_{\mathbf{c}_i}(\tau_{\min}, \chi)$, $\underline{p}_i = \frac{\partial \mathbf{M}_{\mathbf{c}_i}}{\partial t}(\tau_{\min}^+, \chi)$, $\overline{n}_i = \mathbf{M}_{\mathbf{c}_i}(\tau_{\max}, \chi)$, $\overline{p}_i = \frac{\partial \mathbf{M}_{\mathbf{c}_i}}{\partial t}(\tau_{\max}^-, \chi)$
    **if** $n_i \leq \tilde{h}(\tau_{\min})$ **then**
      $\theta$ is the unique solution of $\mathbf{M}_{\mathbf{c}_i}(t, \chi) = \tilde{h}(t)$ on $[\tau_{\min}, \tau_{\max}]$.
      **if** $\mathbf{M}_{\mathbf{c}_i}(\theta, \chi) = \mathbf{M}_{\mathbf{c}}(\theta, \chi)$, $T = T \cup \{\theta\}$
    **else if** $\underline{n}_i + \underline{p}_i(\tau_{\max} - \tau_{\min}) \leq \tilde{h}(\tau_{\max})$ **and** $\overline{n}_i - \overline{p}_i(\tau_{\max} - \tau_{\min}) \leq \tilde{h}(\tau_{\min})$ **then**
      $t^*$ is the largest minimizer of $\mathbf{M}_{\mathbf{c}_i}(t, \chi) - \tilde{h}(t)$ in $[\tau_{\min}, \tau_{\max}]$, $\delta = \mathbf{M}_{\mathbf{c}_i}(t^*, \chi) - \tilde{h}(t^*)$
      **if** $\delta \leq 0$ **then**
        $\theta$ is the unique solution of $\mathbf{M}_{\mathbf{c}_i}(t, \chi) = \tilde{h}(t)$ in $[t^*, \tau_{\max}]$
        **if** $\mathbf{M}_{\mathbf{c}_i}(\theta, \chi) = \mathbf{M}_{\mathbf{c}}(\theta, \chi)$, $T = T \cup \{\theta\}$
      **end if**
    **end if**
  **end for**
  $\tau_{\max} = \tau_{\min}$, $\tau_{\min} = \max\{[0, \tau_{\max}) \cap \mathcal{K}\}$
**end while**

*Proof:* According to the feasibility conditions (13), $\min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma)(\tilde{\tau}_1, \chi) \geq \tilde{h}(\tilde{\tau}_1)$. We denote by $\mathbf{c}_i$ the value condition $i$, *i.e.* $\mathbf{c}_i = \mathcal{M}_{0_i}$ if $i \leq I_0$ and $\mathbf{c}_i = \gamma_{i - I_0}$ if $i > I_0$. We define $\mathbf{c} = \min_i \mathbf{c}_i$. We search for the largest $t \leq \tilde{\tau}_1$ such that $\exists i \in \{1, \ldots, I_0 + I_\gamma\}$ satisfying $\mathbf{M}_{\mathbf{c}_i}(t, \chi) = \tilde{h}(t) = \mathbf{M}_{\mathbf{c}}(t, \chi)$. If no such $t$ exists, there is no solution to (12) in $[0, \tilde{\tau}_1]$, otherwise, this value of $t$ is also the largest solution of (12) in $[0, \tilde{\tau}_1]$.

Let $T$ represent the current set of solutions of (12), initialized to the empty set. We initialize $\tau_{\max} = \tilde{\tau}_1$. The algorithm iteratively updates $\tau_{\max}$ such that, if $T = \emptyset$, there is no solution of (12) in $[\tau_{\max}, \tilde{\tau}_1]$, otherwise the algorithm terminates and the largest element of $T$ is the largest solution of (12) in $[0, \tilde{\tau}_1]$. More precisely, $\forall t \in [\tau_{\max}, \tilde{\tau}_1], \forall i \in \{1, \ldots, I_0 + I_\gamma\}$, $\mathbf{M}_{\mathbf{c}_i}(t, \chi) \geq \tilde{h}(t)$.

This condition is true as we initialize $\tau_{\max}$ because of the compatibility condition (13). Each component $\mathbf{M}_{\mathbf{c}_i}$ can be computed explicitly [6] and we can define three domains in which the solution has a specific analytical expression. For

$i \in \{1, \ldots, I_0 + I_\gamma\}$, we define $\underline{\kappa}_i \leq \kappa_i^1 \leq \kappa_i^2$ corresponding to the boundaries of the three different domains in $x = \chi$. We have $\mathbf{M}_{\mathbf{c}_i}(t, \chi) = +\infty$ if and only if $t \leq \underline{\kappa}_i$ and $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi)$ is affine on the interval $[\kappa_i^1, \kappa_i^2]$. For a given $\tau_{\max}$, we define $\tau_{\min}$ as $\tau_{\min} = \max\{[0, \tau_{\max}) \cap \mathcal{K}\}$.

The solution $\mathbf{M}_{\mathbf{c}_i}$ associated with the convex target function $\mathbf{c}_i$ is convex [6] which implies the convexity of $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi)$. We define $\underline{n}_i = \mathbf{M}_{\mathbf{c}_i}(\tau_{\min}, \chi)$, $\overline{n}_i = \mathbf{M}_{\mathbf{c}_i}(\tau_{\max}, \chi)$, $\underline{p}_i$ is the right derivative of $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi)$ at $\tau_{\min}$ (denoted $\frac{\partial \mathbf{M}_{\mathbf{c}_i}}{\partial t}(\tau_{\min}^+, \chi)$) and $\overline{p}_i$ is the left derivative of $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi)$ at $\tau_{\max}$ (denoted $\frac{\partial \mathbf{M}_{\mathbf{c}_i}}{\partial t}(\tau_{\max}^-, \chi)$). For $i \in \{1, \ldots, I_0 + I_\gamma\}$, we check the following conditions:

1. *If* $\underline{n}_i \leq \tilde{h}(\tau_{\min})$: The function $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi) - \tilde{h}(t)$ is convex in $[\tau_{\min}, \tau_{\max}]$ as the sum of two convex functions. It is negative at $\tau_{\min}$ and positive at $\tau_{\max}$. The function has a unique zero in $[\tau_{\min}, \tau_{\max}]$, which we add to the set $T$ if $\mathbf{M}_{\mathbf{c}_i}(\theta, \chi) = \mathbf{M}_{\mathbf{c}}(\theta, \chi)$.

2. *If* $\underline{n}_i + \underline{p}_i(\tau_{\max} - \tau_{\min}) \leq \tilde{h}(\tau_{\max})$ *and* $\overline{n}_i - \overline{p}_i(\tau_{\max} - \tau_{\min}) \leq \tilde{h}(\tau_{\min})$: The convex function $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi) - \tilde{h}(t)$ is positive in $[\tau_{\min}, \tau_{\max}]$ if and only if its minimum on this interval is positive. Since the function is convex it has a unique minimum $\delta$ reached on a closed interval and we denote by $t^*$ the upper bound of this interval. If $\delta \leq 0$, there exists a unique zero in the interval $[t^*, \tau_{\max}]$ which we add to the set $T$ if $\mathbf{M}_{\mathbf{c}_i}(\theta, \chi) = \mathbf{M}_{\mathbf{c}}(\theta, \chi)$.

3. *If none of the previous conditions is satisfied, the function* $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi) - \tilde{h}(t)$ *is positive in* $[\tau_{\min}, \tau_{\max}]$: We have $\underline{n}_i > \tilde{h}(\tau_{\min})$ and at least one of the following conditions holds: (1) $\underline{n}_i + \underline{p}_i(\tau_{\max} - \tau_{\min}) > \tilde{h}(\tau_{\max})$ or (2) $\overline{n}_i - \overline{p}_i(\tau_{\max} - \tau_{\min}) > \tilde{h}(\tau_{\min})$. If the first condition holds, the function $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi)$ is convex in $[\tau_{\min}, \tau_{\max}]$ so $\mathbf{M}_{\mathbf{c}_i}(t, \chi) \geq \underline{n}_i + (t - \tau_{\min})\underline{p}_i$. Given that $\underline{n}_i > \tilde{h}(\tau_{\min})$, and $\underline{n}_i + \underline{p}_i(\tau_{\max} - \tau_{\min}) > \tilde{h}(\tau_{\max})$, the linear function $t \mapsto \underline{n}_i + \underline{p}_i(t - \tau_{\min})$ is greater than $\tilde{h}$ at $t = \tau_{\min}$ and $t = \tau_{\max}$ and thus, in the entire interval $[\tau_{\min}, \tau_{\max}]$. It implies that $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi) - \tilde{h}(t)$ is positive in $[\tau_{\min}, \tau_{\max}]$. If the second condition holds, a similar reasoning implies that $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi) - \tilde{h}(t)$ is positive in $[\tau_{\min}, \tau_{\max}]$ which concludes the proof.

*Stopping condition:* After checking conditions 1, 2 and 3 above for all $i$, there are two possible cases:

- If $T = \emptyset$, the function $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi) - \tilde{h}(t)$ is positive in $[\tau_{\min}, \tau_{\max}]$ for all $i$. We set $\tau_{\max} = \tau_{\min}$ and keep the property that $\mathbf{M}_{\mathbf{c}_i}(t, \chi) - \tilde{h}(t) \geq 0, \forall t \in [\tau_{\max}, \tilde{\tau}_1]$. We update $\tau_{\min} = \max\{[0, \tau_{\max}) \cap \mathcal{K}\}$ and iterate.
- If $T \neq \emptyset$, its largest element is the largest solution of (12) in the interval $[\tau_{\min}, \tau_{\max}]$ and thus in the interval $[0, \tilde{\tau}_1]$. We terminate the algorithm. ■

*Remark 1 (Analytical solution of $\tau_1$):* In the intervals $[\tau_{\min}, \tau_{\max}]$ such that $\tau_{\min} \geq \kappa_i^1$ and $\tau_{\max} \leq \kappa_i^2$, the function $t \mapsto \mathbf{M}_{\mathbf{c}_i}(t, \chi)$ is affine. Its minimum or zeros are computed by solving a scalar linear equation.

## V. Practical Implementation

We are given a concave Hamiltonian $\psi$, piecewise affine upstream and initial boundary conditions $\gamma$ and $\mathcal{M}_0$, which simulate value conditions of a road segment. We illustrate the reason why the resolution of Problem 1 is important to reconstruct capacity reductions in flow networks. We use Algorithm 1 to solve the reconstruction problem and compute the corresponding solution of Problem 1.

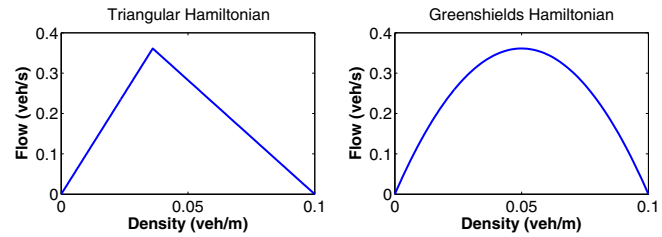


Fig. 1. Concave Hamiltonians $\psi$ used in the numerical simulations. In the context of traffic flow, they represent the empirical relation between flow and density. **Left**: Triangular Hamiltonian, parameterized by the free flow speed ($\nu^\flat = 10$ m/s), the capacity ($q_{\max}$ = 1300 veh/h) and the maximum density ($\rho_{\max} = 1/10$ veh/m). **Right**: Greenshields Hamiltonian, parameterized by the capacity ($q_{\max}$ = 1300 veh/h) and the maximum density ($\rho_{\max} = 1/10$ veh/m).

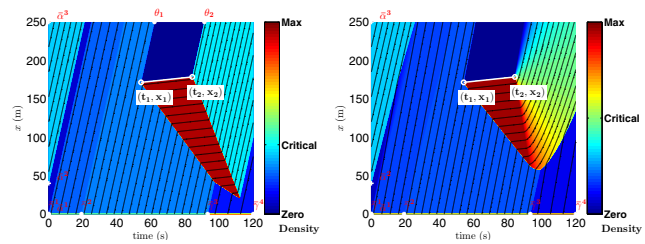


Fig. 2. Solution of (1) given initial and upstream piecewise affine boundary conditions and one affine internal value condition between $(t_1, x_1)$ and $(t_2, x_2)$. **Top:** Solution computed for a triangular Hamiltonian. **Bottom:** Solution computed for a Greenshields Hamiltonian.

### A. Experimental setting

We are given piecewise affine initial and upstream boundary conditions $\mathcal{M}_{0_i}, i \in \{1, \ldots, I_0\}$ and $\gamma_j, j \in \{1, \ldots, I_\gamma\}$, which are generated randomly for the numerical example of interest. In the context of traffic flows, this corresponds to information on vehicle counts at the upstream boundary of the road segment. We also consider an affine internal value condition $\mu$ that satisfies the compatibility conditions with the initial and upstream boundary conditions and represents a vehicle reporting information on a portion of its trajectory, during which its speed is considered constant. The computations are performed for two concave Hamiltonians (illustrated Figure 1), which are commonly used in transportation engineering [12], [17]. The numerical solution is computed using a toolbox developed for Matlab [13], which evaluates the exact solution numerically at any point $(t, x)$ with a low computational cost.

### B. Solution with piecewise affine initial and upstream boundary conditions and one affine internal condition

We compute the solution of (1) with the prescribed piecewise affine initial and upstream boundary conditions and the affine internal condition as the minimum of $\mathbf{M}_{\mathcal{M}_0}$, $\mathbf{M}_\gamma$ and $\mathbf{M}_\mu$ [6]. This solution does not take into account the fact

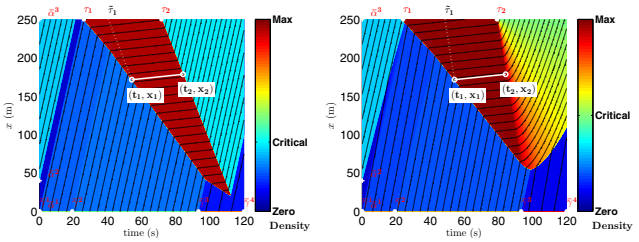


Fig. 4. Solution of the reconstruction problem 1 given initial and upstream piecewise affine boundary conditions and one affine internal value condition between $(t_1, x_1)$ and $(t_2, x_2)$. **Top:** Solution computed for a triangular Hamiltonian. **Bottom:** Solution computed for a Greenshields Hamiltonian.

that the internal value condition results from both the initial, upstream *and* downstream boundary conditions (even though not observed directly), resulting in a domain of null flow and density downstream of the internal value condition between $\theta_1$ and $\theta_2$ (Figure 2).

A strong motivation for solving Problem 1 is the following. Let $\overline{\beta}$ be the value of the solution of (1) in $[0, t_{\max}] \times \{\chi\}$ with the prescribed value conditions $\mathcal{M}_0$, $\gamma$ and $\mu$. The solution of (1) with prescribed value conditions $\mathcal{M}_0$, $\gamma$ and $\overline{\beta}$ leads to a different solution, in particular one which does not coincide with $\mu$, as shown in Figure 3. The motivation is also intuitive in the context of traffic flow engineering, where Figure 2 corresponds to having a vehicle suddenly breakdown when there is no obstacle in front of it. Slow downs are expected to be due to queues caused by downstream capacity reductions.

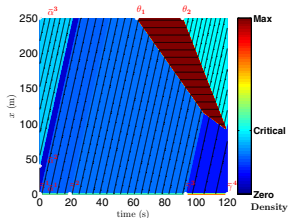


Fig. 3. Solution of (1) with value conditions $\mathcal{M}_0$, $\gamma$ and $\overline{\beta}$.

### C. Resolving the domains of null flow and density

To take into account the fact that the internal condition is not only caused by the initial and upstream conditions but also by the downstream condition, we solve Problem 1 using Algorithm 1, *i.e.*, we reconstruct a downstream boundary condition that "*caused*" the internal value condition. The algorithm computes a solution that represents a constant limitation of the maximum flow for a time interval $[\tau_1, \tau_2]$, as illustrated in Figure 4 for the two concave Hamiltonians. Note that the solution is unique (among the piecewise affine solutions) for an interval $[\tilde{\tau}_1, \tilde{\tau}_2]$ included in $[\tau_1, \tau_2]$ and that other downstream boundary conditions are possible out of this interval.

## VI. Conclusion

We studied a reconstruction problem of downstream boundary conditions from Lagrangian sensing and prescribed upstream and initial conditions, with important applications in flow networks estimation and control. Under compatibility conditions, a downstream boundary condition representing a constant capacity drop can be reconstructed and we present a computationally efficient algorithm that numerically computes the solution.

We discuss the uniqueness of the solution on specific domains, among piecewise affine boundary conditions. The generalization of the algorithm when several internal conditions are given, when the compatibility conditions are not satisfied or when specific conditions (such as periodicity) are imposed on the boundary conditions are the subject of current work.

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